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Knots, Polynomials, and Categorification

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Abstract. These lectures give an introduction to knot polynomials and their categorifications. Topics covered include the Jones and Alexander polynomials, Khovanov homology of links and tangles, HOMFLY-PT homology, and the Λ^k-colored HOMFLY-PT polynomial.

Contents

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1. Prelude: Knots and the Jones polynomial

1.1. Knots

Definition 1.1.1. An *oriented knot* in $\mathbb{R}^3$ is a smooth embedding $K : S^1 \hookrightarrow \mathbb{R}^3$. We say that two knots $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$ are *isotopic* if there is a smooth map $\Phi : S^1 \times [0,1] \to \mathbb{R}^3$ such that $K_t := \Phi|_{S^1 \times t}$ is a knot for each t.

We declare two knots to be equivalent if they are isotopic, and leave it to the reader to check that isotopy is indeed an equivalence relation. Since any two orientation preserving homeomorphisms of S^1 are isotopic, the equivalence class of knot is determined by its oriented image in $\mathbb{R}^3$.

To draw pictures of a knot, we consider its image under a linear projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. We say that the image is *generic* if 1) $\pi \circ K : S^1 \to \mathbb{R}^2$ is an immersion, and 2) $\pi \circ K$ is injective except at transverse double points, which we refer to as *crossings*. By a small isotopy, we may arrange that the image of K under any given projection π is generic.

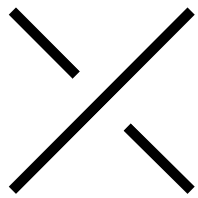
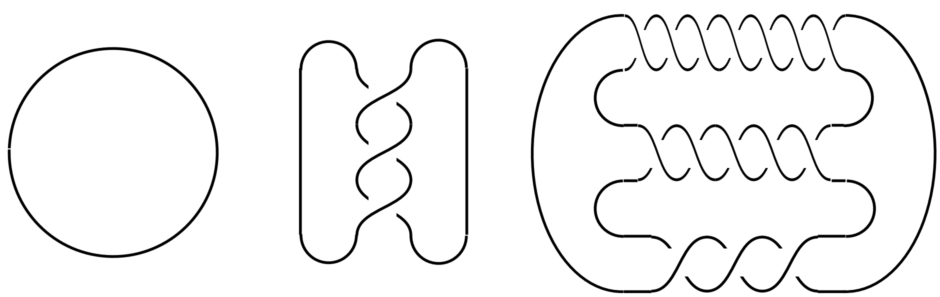


Figure 1.1.2. A crossing

Let us identify the image of the projection π with the xy plane in $\mathbb{R}^3$. If π is generic, the isotopy class of K is determined by the image $\pi \circ K$ together with the relative z-coordinate of the two strands near each double point. The strand with the larger z-coordinate is an *overcrossing*, while the strand with the smaller z-coordinate is an *undercrossing*.

We record this information by briefly picking up our pen near each undercrossing, as shown in the figure. The resulting picture is a *planar diagram* of the knot K. Here are some examples:



Unknot $\bigcirc$ Trefoil T Pretzel knot $P(-3, 5, 7)$

Figure 1.1.3. Examples of Knots

We declare two planar diagrams to be equivalent if there is an orientation preserving homeomorphism of the plane which carries one to the other. For example, any crossingless planar diagram is equivalent to the standard diagram of the unknot shown above. (This seemingly obvious fact is far from trivial — it is the 2-dimensional Schoenflies theorem.)

Note that any given knot will have many inequivalent planar diagrams. For example, the first three diagrams in Figure 1.1.4 all represent the unknot.

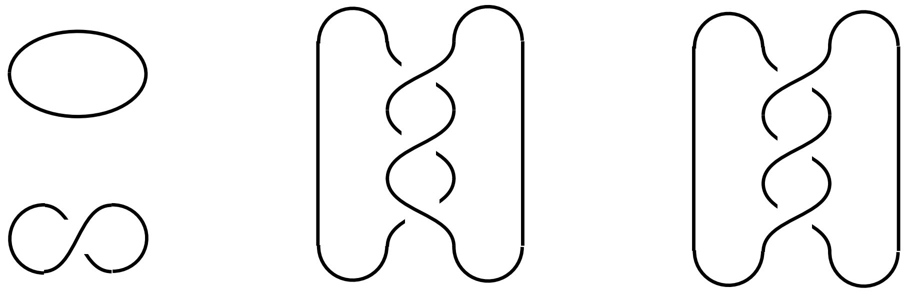


Figure 1.1.4. One of these knots is not like the others.

Definition 1.1.5. Planar diagrams D_1 and D_2 are related by a *Reidemeister move* if there are embeddings $\varphi_1, \varphi_2 : B^2 \hookrightarrow \mathbb{R}^2$ such that

1) $D_1 \cap (\mathbb{R}^2 - \varphi_1(B^2)) = D_2 \cap (\mathbb{R}^2 - \varphi_2(B^2))$
2) $\varphi_1^{-1}(D_1)$ and $\varphi_2^{-1}(D_2)$ are one of the three pairs of diagrams shown in Figure 1.1.6.

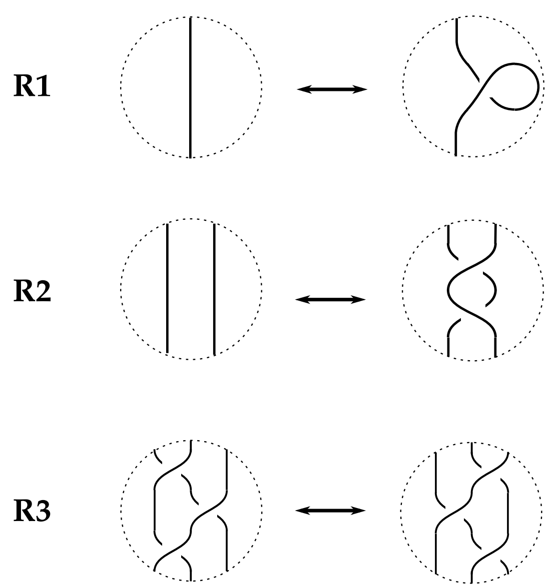


Figure 1.1.6. Reidemeister moves

The reader should convince themselves that diagrams related by a Reidemeister move are indeed isotopic. For example, R1 corresponds to giving a strand of the diagram a half twist; R2 to sliding one strand of the diagram over another, and R3 to sliding a crossing under a strand. As a handy mnemonic, note that the number of the move is equal to the number of distinct strands involved in it.

Theorem 1.1.7 (Reidemeister)**.** *If two planar diagrams represent the same knot, they are related by a sequence of Reidemeister moves.*

1.2. Generalizations One way to generalize Definition 1.1.1 is to change the domain of the embedding.

Definition 1.2.1. We define an *oriented* n*-component link* in $\mathbb{R}^3$ to be a smooth embedding $\amalg^n S^1 \to \mathbb{R}^3$. Two such links L_0 and L_1 are isotopic if there is a smooth map $\Phi : (\amalg^n S^1) \times [0,1] \to \mathbb{R}^3$ such that $L_t := \Phi_{(\amalg^n S^1)\times t}$ is a link for all $t \in [0,1]$.

A few examples are shown in Figure 1.2.2.

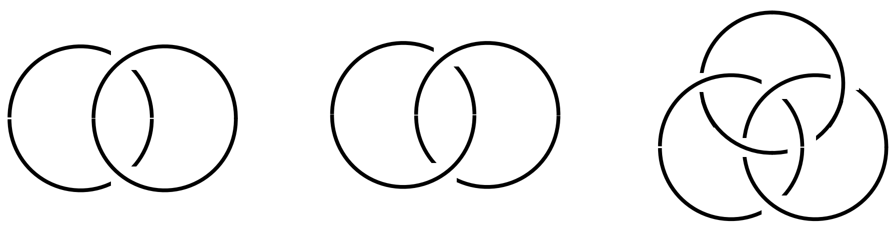


Figure 1.2.2. Examples of Links

Links have planar diagrams just like knots do, and satisfy an analog of Theorem 1.1.7. We can also vary the target of the embedding. If Y is a smooth 3-manifold, an oriented knot in Y is a smooth embedding $K : S^1 \hookrightarrow Y$. Viewing S^3 as the one-point compactification of $\mathbb{R}^3$, we see that every knot in $\mathbb{R}^3$ gives rise to a knot in S^3. Conversely, if K is a knot in S^3 and p is a point, it's easy to see that K is isotopic to a knot which does not contain p. Hence every knot in S^3 arises in this way. Similarly, if two knots are isotopic in S^3, they are isotopic in $\mathbb{R}^3$. (The trace of the isotopy is 2-dimensional, so the isotopy can itself be isotoped to avoid p.) In summary, there is a natural bijection between the set of knots in $\mathbb{R}^3$ and the set of knots in S^3.

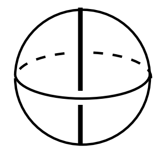
1.3. New knots from old

Definition 1.3.1. If $K \hookrightarrow \mathbb{R}^3$ is a knot, its *mirror knot* $\overline{K}$ is $\rho(K)$, where $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orientation reversing homeomorphism.

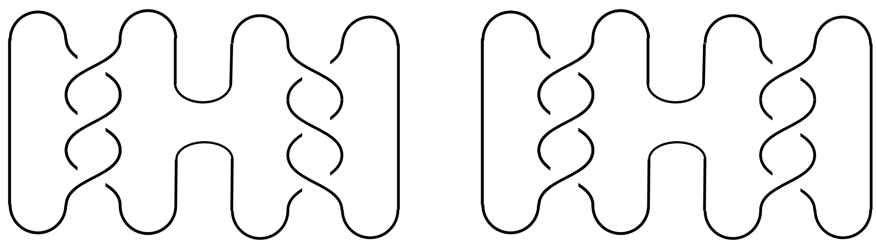
Note that any two such ρ are isotopic, so it doesn't matter which one we pick. Taking r to be reflection in the xy-plane, we see that if D is a diagram of K, the diagram $\overline{D}$ obtained by switching all overcrossings for undercrossings is a diagram of $\overline{K}$.

Definition 1.3.2. Choose orientation preserving embeddings $i_1, i_2 : \mathbb{R}^3 \to \mathbb{R}^3$ whose images are disjoint balls in $\mathbb{R}^3$. If L_1 and L_2 are oriented links in $\mathbb{R}^3$, their *disjoint union* is the union of $i_1(L_1)$ and $i_2(L_2)$.

All orientation preserving embeddings $\mathbb{R}^3 \to \mathbb{R}^3$ are isotopic, so again, the exact choice of the maps i_1 and i_2 is irrelevant.



Finally, we describe the connected sum operation. If K_1, K_2 are knots in S^3, choose balls $B_1, B_2 \subset S^3$ such that $B_i \cap K_i$ is a single unknotted arc as shown in the figure to the right. Let $Y_i = S^3 \setminus \mathrm{int}(B_i)$. The orientation on $K_i \cap B_i$ induces an orientation on $K_i \cap \partial B_i$; one of the two points is positively oriented, the other negatively. Let $\varphi : \partial B_1 \to \partial B_2$ be reflection in the vertical direction of the figure (so $\varphi(K_1 \cap B_1) = K_2 \cap B_2$, but with the orientation reversed.)



Square Knot $T\#\overline{T}$ Granny Knot $T\#T$

Figure 1.3.3. Connected sums

Definition 1.3.4. The connected sum $K_1\#K_2$ is the knot obtained by taking

$$(K_1 \cap Y_1) \cup_\varphi (K_2 \cap Y_2) \subset Y_1 \cup_\varphi Y_2 = S^3$$

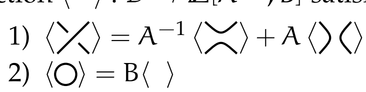
and smoothing.

Again, this definition does not depend on the exact choice of the balls B_1 and B_2. Note that it does depend on the orientation on K_1 and K_2 (just as with connected sum of manifolds). If we want to take the connected sum of two links, we must specify the components we are summing along.

1.4. The Jones polynomial It would be nice to have some way of telling when two diagrams represent different links. For example, we'd like to be able to say that the rightmost diagram in Figure 1.1.4 is not the unknot. Our experience with shoelaces and electrical cables suggests that this is the case, but how can we prove it?

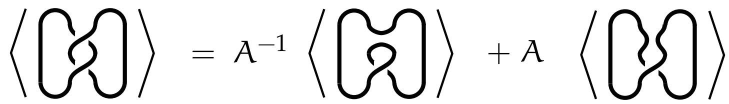
One approach is suggested by Theorem 1.1.7. Let $\mathcal{D}$ be the set of all link diagrams up to isotopy, let $\mathcal{K}$ the set of all links in $\mathbb{R}^3$, and let X be some other set. If we can define a function $I : \mathcal{D} \to X$ such that $I(D) = I(D')$ whenever D and D' are related by a Reidemeister move, I will descend to a function $I : \mathcal{K} \to X$. In other words, I is a *link invariant*.

We can define a link invariant with the help of the *Kauffman bracket*, which is a function $\langle \cdot \rangle : \mathcal{D} \to \mathbb{Z}[A^{\pm 1}, B]$ satisfying the local relations

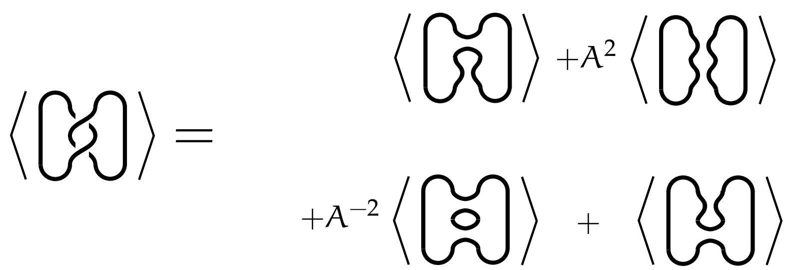


together with the normalization $\langle \emptyset \rangle = 1$, where $\emptyset$ denotes the empty diagram.

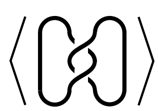
Relation 1) should be interpreted in the following way: if we have any three planar diagrams which agree outside of a disk $B^2 \subset \mathbb{R}^2$ and whose intersections with B^2 are the three diagrams shown, then their brackets satisfy the relation. For example, applying this relation to the upper crossing in our diagram of the Hopf link gives



Applying the same relation to the lower crossing, we get



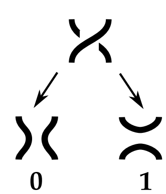
Applying the second relation, which allows us to remove a crossingless circle, we get



$$= A^{-2}B^2 + 2B + A^2B^2$$

The procedure above can be applied to any planar diagram D. By turning our heads, we can arrange that any unoriented crossing in D looks like the one on the left-hand side of relation 1). Hence by applying relation 1) repeatedly, we can reduce $\langle D \rangle$ to a sum of terms involving the brackets of planar diagrams with no crossings, which can then be simplified using relation 2).

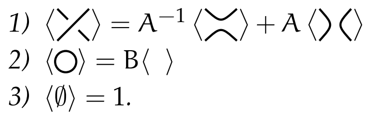
We can formalize this a bit by saying that each crossing has two *resolutions*, which we call the *0 resolution* and *1 resolution*, as shown in the figure to the right. If D has n crossings, there will be 2^n ways to resolve all n, and these 2^n diagrams are in bijection with the vertices of the n-dimensional cube $[0,1]^n$. If $D_{\mathbf{v}}$ denotes the crossingless planar diagram at vertex $\mathbf{v}$, then



$$\langle D \rangle = \sum_{\mathbf{v}} A^{n-2|\mathbf{v}|} B^{|D_{\mathbf{v}}|} \tag{1.4.1}$$

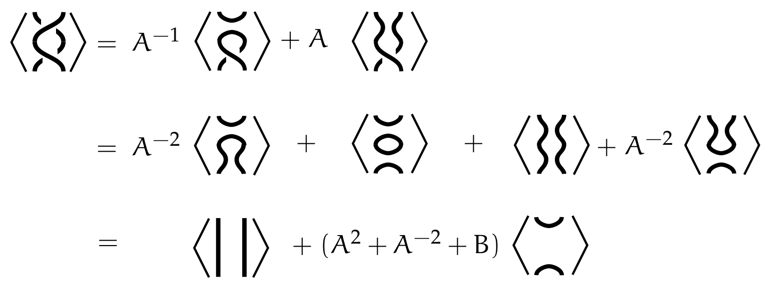
where $|\mathbf{v}|$ denotes the sum of the coefficients of $\mathbf{v}$, and $|D_{\mathbf{v}}|$ denotes the number of components of $D_{\mathbf{v}}$. To summarize, we have the following

Lemma 1.4.2. *There is a unique function* $\langle \cdot \rangle : \mathcal{D} \to \mathbb{Z}[A^{\pm 1}, B]$ *satisfying the local relations*



Proof. We've shown above that relations 1)-3) imply that $\langle D\rangle$ is given by equation (1.4.1). Conversely, if D is any planar diagram, we may define $\langle D\rangle$ by equation (1.4.1). It is then easy to see that D satisfies relations 1) and 2). □

Next, we consider how the bracket changes under Reidemeister moves. Skipping over R1 for the moment, we consider the effect of the R2 move. We compute

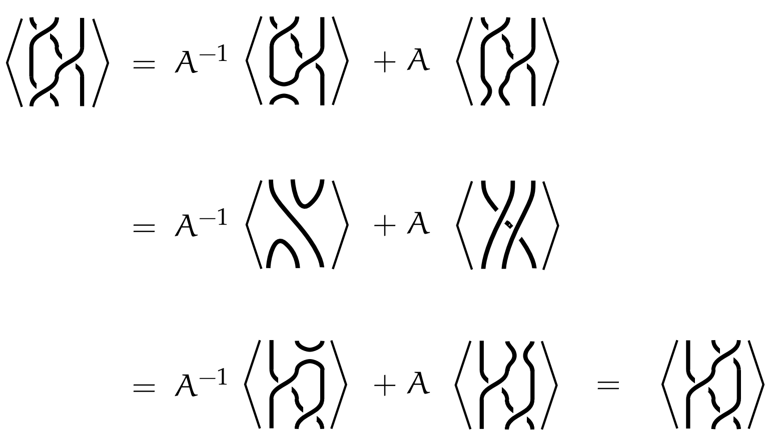


and see that in order for the bracket to be invariant under R2, we must set

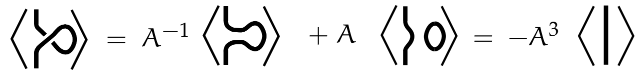
$$\boxed{B = -A^{-2} - A^{2}.}$$

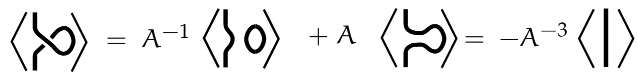
Although this is sufficient to ensure invariance under the R2 move, it's not very promising. We still have two Reidemeister moves to go, and have already eliminated one of our two variables. Nevertheless, we press on and consider the R3 move.

At first, it seems like we'll have to resolve three crossings, resulting in eight different diagrams and a very messy calculation. In fact, since we've already proved that the bracket is invariant under R2, it's enough to resolve one:



It follows that the bracket is invariant under the R3 move without any further specialization. It remains only to consider the R1 move. We compute





Disappointingly, the we find that the bracket is **not** invariant under the R1 move, and hence **not** a link invariant. But it's very close — close enough, in fact, that there is a fix for our problem.

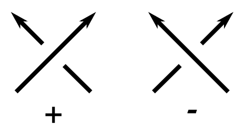


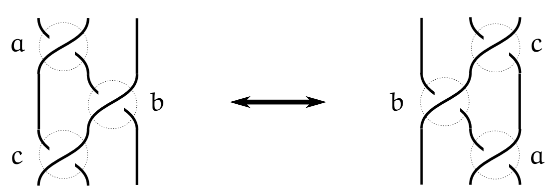
Figure 1.4.3. Positive and negative crossings

The Fix: So far, we have been thinking about unoriented link diagrams. If we pay attention to the orientations, it is no longer true that every crossing looks like the one in Figure 1.1.2. Instead there are two possible types, which we refer to as *positive* and *negative* crossing. These are shown in the figure above. If D is a planar diagram, we write $n_\pm(D)$ for the number of positive/ negative crossings of D, and define the *writhe* of D to be

$$w(D) = n_+(D) - n_-(D).$$

Lemma 1.4.4. *The writhe is invariant under Reidemeister moves 2 and 3.*

Proof. The two additional crossings in the right-hand diagram for R2 will always have opposite signs, no matter how we orient the strands. Hence the net change to the writhe is 0. For the R3 move, consider the figure below:



The crossings labelled a will have the same sign on both sides of the figure, regardless of the orientation of the strands. Similarly for the crossings b and c, so the writhe will be the same. □

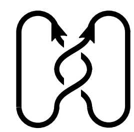
On the other hand, an R1 move will either increase or decrease the writhe by 1. We can use this to counteract the change in the Kauffman bracket under an R1 move.

Definition 1.4.5. If D is an oriented link diagram, we define its *unreduced Jones polynomial* to be

$$\overline{V}(D) = (-A^3)^{-w(D)}\langle D\rangle.$$

Theorem 1.4.6. *The unreduced Jones polynomial is an invariant of oriented links.*

Proof. We have already seen that both the writhe and the bracket are invariant under R2 and R3, so $\overline{V}(D)$ is invariant under R2 and R3. If D and D' are the left and right-hand figures in the diagram of the R1 move in Figure 1.1.6, then $w(D') = w(D) + 1$ and $\langle D' \rangle = -A^3 \langle D' \rangle$. It follows that $\overline{V}(D') = \overline{V}(D)$. □



Example 1.4.7. With this definition, we have $V(\emptyset) = 1$, where $\emptyset$ denotes the empty link, and $\overline{V}(\bigcirc) = -A^{-2} - A^2$. More generally, $\overline{V}(\bigcirc^n) = (-A^{-2} - A^2)^n$, where $\bigcirc^n$ denotes the n-component unlink. If H_+ is the positively oriented Hopf link (shown to the right), we have already computed that

$$\langle H_+ \rangle = B(A^{-2}B + 2 + A^2 B) = (-A^{-2} - A^2)(-A^{-4} - A^4) = A^{-6} + A^{-2} + A^2 + A^6,$$

so

$$\overline{V}(H_+) = A^{-12} + A^{-8} + A^{-4} + 1.$$

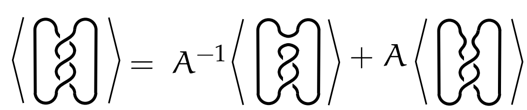
We deduce that the Hopf link is not the unlink.

Normalizations If D is a nonempty link diagram, then every complete resolution $D_{\mathbf{v}}$ will have at least one component. It follows that every term of the sum in equation (1.4.1) is divisible by $B = \overline{V}(\bigcirc)$, so $\overline{V}(D)$ is divisible by $\overline{V}(\bigcirc)$.

Definition 1.4.8. A nonempty link L has a *reduced Jones polynomial* $V(L)$ given by

$$V(L) = \overline{V}(L)/\overline{V}(\bigcirc).$$

Example 1.4.9. Let T be the positive trefoil knot, as shown in Figure 1.1.3. We compute the bracket as in the figure above.



$$= A^{-1}(-A^{-3})^2 B + AB(-A^{-4} - A^4)$$
$$= (A^{-7} - A^{-3} - A^5)B$$

Since $w(D) = 3$, we get $V(T) = -A^{-16} + A^{-12} + A^{-4}$. We conclude that the trefoil is not the unknot.

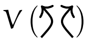
Although the variable A is most natural for the Kauffman bracket, when we work with the Jones polynomial, we will generally use the variable

$$\boxed{q = -A^{-2}.}$$

We have $\overline{V}(\bigcirc^n) = (q + q^{-1})^n$, $V(H_+) = q + q^5$, and $V(T) = q^2 + q^6 - q^8$ in this normalization. When we want to emphasize the variable being used, we put the link in the subscript, writing $V_L(q)$ for what we previously denoted by $V(L)$.

As the examples above suggest, $V_L(q) \in \mathbb{Z}[q^{\pm 1}]$ for any link L, although *a priori* we might expect to see terms involving $(-q)^{1/2}$. One way to prove this is via the following

Proposition 1.4.10. *The Jones polynomial satisfies the* oriented skein relation

$q^2 V$ () $- q^{-2} V$ () $= (q - q^{-1}) V$ () .

As usual, this is to be interpreted as a local relation. For example, applying the skein relation to a crossing in our standard diagram for the trefoil gives

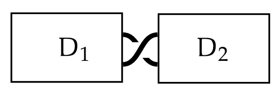
$$q^2 V(\bigcirc) - q^{-2} V(T) = (q - q^{-1}) V(H_+).$$

1.5. Connections and Further Reading The Kauffman bracket was introduced in a classic paper by Kauffman [37], which is well worth reading. This paper also contains an important early application of the Jones polynomial which we have not discussed: it gives a lower bound on the crossing number of a link.

Theorem 1.5.1. *Let* $M(L)$ *and* $m(L)$ *be the maximum and minimum degrees of the Laurent polynomial* $V_L(q)$*. If* D *is any planar diagram representing* L*, then*

$$2c(D) \geqslant M(L) - m(L).$$

If D *is reduced and alternating then equality holds. In contrast, if* D *is non-alternating, then the inequality is strict.*



Here a diagram D is *reduced* if it does not contain a crossing of the form shown in the figure below. D is *alternating* if crossings alternate between over and under as we traverse any component. For example, the diagrams of the trefoil and Borromean rings in Figures 1.1.3 and 1.2.2 are both alternating, while the pretzel knot $P(-3, 5, 7)$ is not.

Theorem 1.5.1 resolved one of the oldest open problems in knot theory, due to Peter Guthrie Tait:

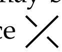
Corollary 1.5.2 (First Tait Conjecture)**.** *Let* D *be a reduced alternating diagram. If* D' *is any other diagram representing the same link, then* $c(D) \leqslant c(D')$*.*

In particular, a reduced alternating diagram with > 0 crossings is always knotted.

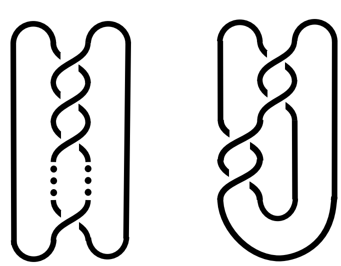
Our strategy of checking invariance under the Reidemeister moves is an effective way of proving that some quantity is a link invariant, but it is a terrible way of finding such invariants to start with. The definition of the Jones polynomial we have given is completely elementary, but remained undiscovered for 100 years after mathematicians first started thinking about knots. Jones arrived at hs original definition [36] by thinking about something entirely different — representations of von Neumann algebras. We'll return to the representation-theoretic approach in lecture 4, when we talk about the HOMFLY-PT polynomial.

After Jones's discovery, Witten realized that the Jones polynomial should fit into a much broader theory of invariants of 3-manifolds defined using Chern-Simons theory. His paper [83] launched an entire industry devoted to the study of these *quantum invariants*, both in physics and mathematics. We'll discuss these more general invariants in lecture 6, but for now it is worth knowing that Witten's approach assigns a polynomial invariant of knots to each complex Lie algebra $\mathfrak{g}$ equipped with a representation V; the Jones polynomial corresponds to the vector representation of $\mathfrak{sl}_2$.

Exercises

1. If K is a knot in S^3, show that $H_1(S^3 - K) = \mathbb{Z}$ and that $H_*(S^3 - K) = 0$ for $* > 1$. More generally, if $L \subset S^3$ is an n-component link, show that $H_1(S^3 - L) = \mathbb{Z}^n$.
2. Suppose $K \subset \mathbb{R}^3$, and let $f : K \to \mathbb{R}$ be projection on the z-coordinate. Show that if f has a single local maximum on K, then K is unknotted. Deduce that any knot diagram may be unknotted by a sequence of crossing changes, in which we replace  by  while leaving the rest of the diagram unchanged.
3. Suppose D is an oriented 2-component link diagram, and let $n^*_\pm(D)$ be the number of positive/negative crossings in which the two strands belong to *different* components of L. Show that the *linking number* $\mathrm{lk}(D)$ defined by setting $\mathrm{lk}(D) = \frac{1}{2}[n^*_+(D) - n^*_-(D)]$ is invariant under all three Reidemeister moves, and hence an invariant of L.
4. Suppose that L is as in the previous exercise, and that L_1 and L_2 are its components. If $S \subset \mathbb{R}^3$ is an oriented embedded surface with $\partial S = L_1$, show that $\mathrm{lk}(L)$ is the intersection number $L_2 \cdot S$
5. Show that $V_{\overline{L}}(q) = V_L(q^{-1})$. Deduce that T and $\overline{T}$ are not equivalent.
6. Show that $V(K_1 \# K_2) = V(K_1)V(K_2)$. Deduce that the square knot $T\#\overline{T}$ and the granny knot $T\#T$ are not equivalent.
7. Use the Kauffman bracket to compute the Jones polynomial of the $(2,n)$ torus knots and of the figure-eight knot, which are shown below. (For something harder, but still doable by hand, try computing $V(P(-3,5,7))$.)
8. Prove the skein relation of Proposition 1.4.10. Deduce that $\overline{V}_L(1) = 2^n$, where n is the number of components of L.

(n crossings)



$T(2,n)$ Figure-eight

2. The Alexander Polynomial

In this lecture, we'll discuss another polynomial invariant of knots, known as the Alexander polynomial. This polynomial is much older than the Jones polynomial, and there's a good reason it was discovered first. It can be derived from a standard invariant of algebraic topology — the fundamental group.

2.1. The knot group If K is a knot in S^3, we let $M_K = S^3 - K$ be its complement.

Lemma 2.1.1. *If K_0 and K_1 are isotopic knots in S^3, then their complements M_{K_0} and M_{K_1} are orientation preserving homeomorphic.*

Proof. (Sketch) If $\Psi : S^1 \times [0,1] \to S^3$ is the isotopy, then we can define a time-dependent vector field $\mathbf{v}_t$ on the image $\Psi(S^1, t)$ by

$$\mathbf{v}_t|_{\Psi(\theta,t)} = d\Psi\left(\frac{\partial}{\partial t}|_{(\theta,t)}\right).$$

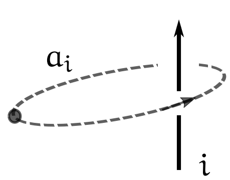
We can extend $\mathbf{v}_t$ to a time-dependent vector field $\mathbf{v}_t$ on all of S^3. This is best done by thinking of $\mathbf{v}_t$ as a vector field on the embedded cylinder

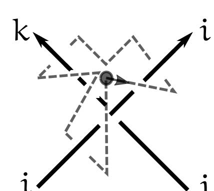
$$\{(\Psi(\theta,t),t) \,|\, (\theta,t) \in S^1 \times [0,1]\} \subset S^3 \times [0,1].$$

If $\psi_t : S^3 \to S^3$ is the flow defined by $\mathbf{v}_t$, then $\psi_1 : S^3 \to S^3$ is an orientation-preserving diffeomorphism satisfying $\psi_1(K_0) = K_1$. $\square$

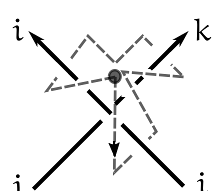
As a consequence, any topological invariant of the knot complement M_K is an invariant of K. The most obvious one to try is the ordinary homology $H_*(M_K)$, but this turns out not to be so useful as an invariant: $H_1(M_K) \simeq \mathbb{Z}$ for every knot K, and $H_*(M_K) = 0$ for $* > 1$. (Lecture 0, Exercise 1). In contrast, $\pi_1(M_K)$ turns out to be a much better invariant. We sketch a method for computing it from a diagram D of K.

Definition 2.1.2. If D is an oriented planar diagram, an arc of D is a segment of D between two undercrossings. The *Wirtinger presentation* associated to D has generators a_i, where $i \in \{\text{arcs of } D\}$ and relations w_c, where $c \in \{\text{crossings of } D\}$. Here we take our basepoint $*$ to lie above the plane of the paper, and a_i to be a loop which runs down from the basepoint, around arc i, and then back up to the basepoint, as shown in the figure on the left below. The orientation of the loop a_i and the arc i should should satisfy the right-hand rule as shown. The relations w_c associated to positive and negative crossings are shown in the two figures on the right.





$a_j = a_i a_k a_i^{-1}$

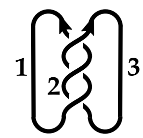


$a_j = a_i^{-1} a_k a_i$

The reader should take a moment to convince her/himself that these relations hold, as illustrated in the figure. Of course, we should really prove that the a_i generate $\pi_1(M_K)$, and that there are no more relations. One approach to this (sketched in the exercises) is to find a handle decomposition of $S^3 - \nu(K)$.

Example 2.1.3. If we use the standard diagram of the trefoil shown to the right, the Wirtinger presentation is

$$\pi_1(M_T) = \langle a_1, a_2, a_3 \mid a_1 = a_3^{-1}a_2a_3, a_3 = a_2^{-1}a_1a_2, a_2 = a_1^{-1}a_3a_1 \rangle.$$



which we rewrite as

$$\pi_1(M_T) = \langle a, b, c \mid a = c^{-1}bc, c = b^{-1}ab, b = a^{-1}ca \rangle.$$

We can eliminate c using the second relation; when we do so, both new relations we get are equivalent to the single relation $abab^{-1}a^{-1}b^{-1} = 1$. Hence

$$\pi_1(M_T) = \langle a, b \mid abab^{-1}a^{-1}b^{-1} = 1 \rangle.$$

Remark 2.1.4. The simplest topological space with this fundamental group is the finite cell complex consisting of a single 0-cell, two 1-cells a and b, and a single 2-cell attached along the curve $abab^{-1}a^{-1}b^{-1}$. In fact, $S^3 - \nu(T)$ has a handle decomposition consisting of a single 0-handle e_0, two 1-handles a and b, and a 2-handle f_2 attached to the one-skeleton along the word $abab^{-1}a^{-1}b^{-1}$. The proof is sketched in the exercises.

If K and $\overline{K}$ are mirror knots, then $S^3 - K$ is (orientation-reversing) homeomorphic to $S^3 - \overline{K}$. Hence $\pi_1(M_K) \simeq \pi_1(M_{\overline{K}})$, and the fundamental group does not distinguish between mirrors. This seemingly trivial example can be leveraged to give less trivial ones by using connected sums. For example, the complements of $T\#T$ and $T\#\overline{T}$ are not homeomorphic, but they have isomorphic fundamental groups.

However, this is essentially the only way in which things can go wrong. If K is a prime knot in S^3 (that is, it cannot be decomposed as a nontrivial connected sum) and $\pi_1(S^3 - K) \simeq \pi_1(S^3 - K')$, then it can be shown that either $K' = K$ or $K' = \overline{K}$ [26,82]. This is a deep result that combines Waldhausen's work on Haken manifolds [23] with Gordon and Luecke's theorem [26] that knots are determined by their complements.

2.2. The infinite cyclic cover The knot group is a very powerful invariant, but it can be hard to tell whether two finitely presented groups are isomorphic. What we'd like is some invariant of a group which can be derived from a presentation, but is easy to compute. The most obvious such invariant is the abelianization. For example, if we abelianize the presentation of $G = \pi_1(M_T)$ given above, we see that

$$G^{ab} = [a, b \mid a + b + a - b - a - b = 0] = [a, b \mid a = b] \simeq \mathbb{Z}$$

where we've used the square brackets to indicate we're writing a presentation of an abelian group. Of course this is no surprise; the abelianization of $\pi_1(M_K)$ is $H_1(M_K) \simeq \mathbb{Z}$, as you computed in the exercises for section 1.

This may seem like a dead end, but in fact it leads somewhere interesting. Let

$$|\cdot| : \pi_1(M_K) \to H_1(M_K) \simeq \mathbb{Z} = \langle t \rangle$$

be the abelianization map. Corresponding to the diagram

$$\begin{array}{ccc} \ker|\cdot| & \longrightarrow & \pi_1(M_K) \\ & & \big\downarrow |\cdot| \\ & & H_1(M_K) \end{array} \qquad \text{there is a regular covering map} \qquad \begin{array}{c} \overline{M}_K \\ p \big\downarrow \\ M_K \end{array}$$

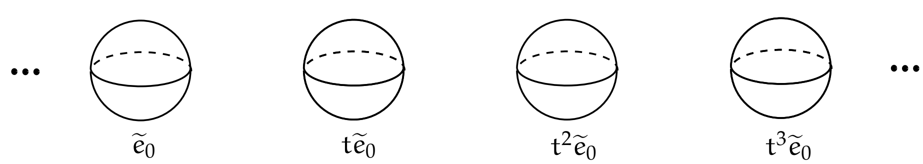
where $\pi_1(\overline{M}_K) = \ker|\cdot|$.

Definition 2.2.1. The covering space $\overline{M}_K$ is the *infinite cyclic cover* of M_K.

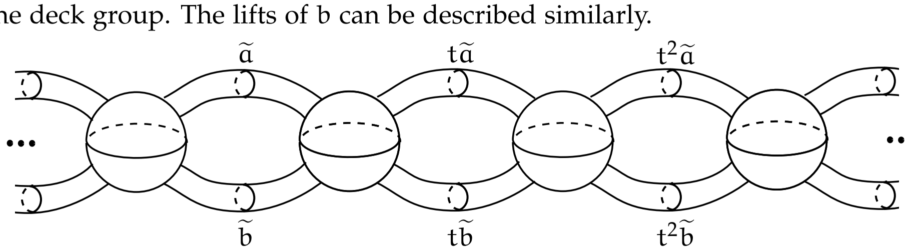
The group of deck transformations is $H_1(M_K) = \mathbb{Z}$, which we take to be generated by $\varphi : \overline{M}_K \to \overline{M}_K$.

Example 2.2.2. Consider the handle decomposition of $S^3 - \nu(T)$ described in Remark 2.1.4. The preimages of the handles under p give a handle decomposition of $\overline{M}_K$. We build this decomposition up one dimension at a time.

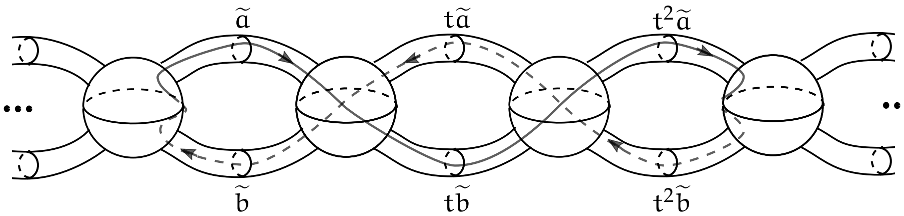
Dimension 0: The preimage of the 0-handle e_0 is an infinite disjoint union of 0-handles $t^k\widetilde{e}_0$, $k \in \mathbb{Z}$. The deck transformation φ acts by $\varphi(t^k\widetilde{e}_0) = t^{k+1}\widetilde{e}_0$.



Dimension 1: Let $\widetilde{a}$ be the lift of a whose foot lies on $\widetilde{e}_0$. The head of $\widetilde{a}$ will lie on $|a|\widetilde{e}_0 = t\widetilde{e}_0$. All the other lifts of $\widetilde{a}$ will be orbits of this lift under the action of the deck group. The lifts of b can be described similarly.



Dimension 2: The attaching circle of the 2-handle f_2 traces out a path given by the word $abab^{-1}a^{-1}b^{-1}$, one of whose lifts is shown in the figure below. The attaching circle of the 2-handle $\widetilde{f}_2$ runs along this lift. All the other lifts (not shown) are orbits of this one under the action of the deck group.



2.3. The Alexander polynomial Now we consider the group $H_1(\overline{M}_K;\mathbb{Q})$. The deck group acts on this group via $t \cdot x = \varphi_*(x)$. Extending linearly, we get an action of the rational group ring $R = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t^{\pm 1}]$ on $H_1(\overline{M}_K;\mathbb{Q})$. In other words, $H_1(\overline{M}_K;\mathbb{Q})$ is an R-module. By the structure theorem for finitely generated modules over a PID, we know that $H_1(\overline{M}_K;\mathbb{Q}) \simeq R^k \oplus R/(p_1) \oplus \dots R/(p_n)$ for some $p_1, \dots p_n \in R$.

Lemma 2.3.1. $H_1(\overline{M}_K;\mathbb{Q})$ *is a torsion module over* R; i.e. $k = 0$.

Proof. The chain complex $C_*(\overline{M}_K;\mathbb{Q})$ is free over R. If we define $N_1 = R/(t-1)$, then we have

$$C_*(\overline{M}_K;\mathbb{Q}) \otimes_R N_1 \simeq C_*(S^3 - K;\mathbb{Q}).$$

Since R is a PID, we can apply the universal coefficient theorem to see that

$$H_1(M_K) = H_1(\overline{M}_K;\mathbb{Q}) \otimes_R N_1 \bigoplus \operatorname{Tor}_R(H_0(\overline{M}_K;\mathbb{Q}), N_1).$$

$\overline{M}_K$ is connected, so $H_0(\overline{M}_K;\mathbb{Q}) \simeq \mathbb{Q}$, and if $x \in H_0(\overline{M}_K;\mathbb{Q})$, $\varphi_*(x) = x$. We conclude that $H_0(\overline{M}_K;\mathbb{Q}) \simeq N_1$ as an R-module, so $\operatorname{Tor}_R(H_0(\overline{M}_K;\mathbb{Q}), N_1) \simeq N_1$ has dimension 1 over $\mathbb{Q}$. On the other hand, we know that $H_1(M_K;\mathbb{Q}) \simeq \mathbb{Q}$, so we must have $H_1(\overline{M}_K;\mathbb{Q}) \otimes_R N_1 = 0$. It follows that $H_1(\overline{M}_K;\mathbb{Q})$ is torsion. □

If N is a finitely-generated torsion module over a PID R, then we can write $N = R/(\alpha_1) \oplus \dots R/(\alpha_n)$.Although this decomposition is not unique, the *order* $\operatorname{ord} N := \alpha_1 \dots \alpha_n$ is well defined up to multiplication by units in R. To indicate this ambiguity, we write $\operatorname{ord} N \sim p(t)$, rather than $\operatorname{ord} N = p(t)$.

If $R = \mathbb{Z}$, so N is a finitely generated abelian group, this definition reduces to the usual notion of the order of a group. Many well-known properties of the order of a group extend to the general situation. Two which we will use are

1) The order is multiplicative. If $N_1 \subset N_2$, we define $[N_2 : N_1] \sim \operatorname{ord} N_2/N_1$. Then if $N_1 \subset N_2 \subset N_2$, we have $[N_3 : N_1] \sim [N_3 : N_2][N_2 : N_1]$.
2) A matrix $A \in M_{n\times n}(R)$ defines a map $A : R^n \to R^n$ for which we have $\operatorname{ord}\operatorname{coker} A \sim [R^n : \operatorname{im} A] \sim \det A$.

Definition 2.3.2. The *Alexander polynomial* of a knot K in S^3 is defined to be $\Delta_K(t) \sim \operatorname{ord} H_1(\overline{M}_K;\mathbb{Q})$.

A priori, $\Delta_K(t)$ is well-defined up to multiplication by units in R; *i.e.* up to multiplication by ct^k, where $c \in \mathbb{Q}$. In fact, as we describe in the next section, this ambiguity can be eliminated to give a well-defined polynomial $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$.

We remark that in contrast to the Jones polynomial, which we defined using a diagram of K, and thus requires the fact that $K \subset S^3$, the definition of $\Delta_K(t)$ only depended on $\pi_1(S^3 - K)$. Hence it extends without change to knots in a homology sphere, and (with a little more work) to knots in an arbitrary 3-manifold Y.

Example 2.3.3. From the handle decomposition in Example 2.2.2, we see that $C_*^{cell}(\overline{M}_T;\mathbb{Q})$ has the form:

$$\begin{array}{ccccc}
C_2^{cell}(\overline{M}_T) & \xrightarrow{\quad d_2 \quad} & C_1^{cell}(\overline{M}_T) & \xrightarrow{\quad d_1 \quad} & C_0^{cell}(\overline{M}_T) \\
\| & \begin{bmatrix} 1+t^2-t \\ t-t^2-1 \end{bmatrix} & \| & \begin{bmatrix} t-1 & t-1 \end{bmatrix} & \| \\
\langle \widetilde{f}_2 \rangle & \longrightarrow & \langle \widetilde{a}, \widetilde{b} \rangle & \longrightarrow & \langle \widetilde{e}_0 \rangle
\end{array}$$

We see that $\ker d_1 = \langle \widetilde{a} - \widetilde{b} \rangle$, while $\operatorname{im} d_2 = \langle (1-t+t^2)(\widetilde{a}-\widetilde{b}) \rangle$. Hence $H_1(\overline{M}_T) = R/(t^2-t+1)$, and $\Delta(T) \sim t^2 - t + 1$.

2.4. Fox calculus The procedure in Examples 2.2.2 and 2.3.3 can be applied to any group presentation. To be specific, suppose

$$G = \langle a_1, \ldots a_m \,|\, w_1, \ldots w_n \rangle$$

is a finitely generated group, and let F be the free group generated by the a_i. We can build a 2-dimensional cell complex X with $\pi_1(X) \simeq G$ by starting with a 0-cell e_0, attaching one 1-cell for each generator a_i, and then attaching one 2-cell along the loop corresponding to each word w_i.

Definition 2.4.1. The *Fox derivative* (or *free derivative*) $d_i : F \to \mathbb{Z}[F]$ is the unique map satisfying the following properties:

1) $d_i a_j = \delta_{ij}$
2) $d_i(ww') = d_i w + [w] d_i w'$

The first property says that d_i behaves like partial derivative with respect to a_i, but the "Leibniz rule" is a bit funny.

In order for the definition to make sense, we must check that there is a unique map d_i satisfying these two properties. If $1 \in F$ is the identity element, then applying the Leibniz rule to the relation $1 \cdot 1 = 1$ gives $d_i(1) = 2d_i(1)$, so $d_i(1) = 0$. Next, by applying d_i to the relation $1 = a_j^{-1} a_j$, we see that

$$d_i a_j^{-1} = -[a_j^{-1}] d_i a_j = -[a_j^{-1}]\delta_{ij}.$$

Since every $w \in F$ can be uniquely expressed as a reduced word in the a_i's and a_i^{-1}'s, it follows that $d_i w$ is determined by property 2). For example,

$$d_a(ca^{-1}b^{-1}a^2ca^{-1}) = -[ca^{-1}] + [ca^{-1}b^{-1}] + [ca^{-1}b^{-1}a] - [ca^{-1}b^{-1}a^2ca^{-1}].$$

We leave it to the reader to check that for any two reduced words w, w' this definition gives $d_i(ww') = d_i w + [w] d_i w'$.

The geometric significance of the Fox derivative is as follows. Let G and X be as above, and suppose $\varphi : G \to G'$ is a surjective homomorphism. Let $\widetilde{X}$ be the corresponding regular cover with deck group G'. The attaching circle w_j lifts up to a closed loop $\widetilde{w}_j$ in the 1-skeleton $\widetilde{X}^{(1)}$.

The cellular chain complex $C_*^{cell}(\widetilde{X})$ is a free module over the group ring $\mathbb{Z}[G']$, generated by lifts of the cells of X. We fix a lift $\widetilde{e}_0$ of the 0-cell, and choose lifts $\widetilde{a}_i$ of the 1-cells by requiring that $\widetilde{a_i}$ has one foot on $\widetilde{e}_0$ and points away from it. Similarly, we choose $\widetilde{w}_j$ so that it is based at $\widetilde{e}_0$.

Lemma 2.4.2. *With these choices we have,*

$$[\widetilde{w}_j] = \sum_{i=1}^{n} \psi(d_i w_j) \cdot [\widetilde{a}_i] \in H_1^{cell}(\widetilde{X}^{(1)}),$$

where $\psi : \mathbb{Z}[F] \to \mathbb{Z}[G']$ *is induced by the composition of* φ *and the projection* $F \to G$.

Proof. Each appearance of a_i in w_j will lift to $g\widetilde{a}_i$ for some $g \in G'$ and hence will contribute $g \cdot [\widetilde{a}_i]$ to $[\widetilde{w}_j]$. To determine g, write $w_j = w a_i w'$, where we've broken out the appearance of a_i we are interested in. Then $g = \psi(w)$, and $g \cdot [\widetilde{a}_i]$ is exactly the term of $d_i w_j$ corresponding to this appearance of a_i. Similarly, an appearance of a_i^{-1} of the form $w = w a_i^{-1} w'$ will contribute $-\psi(w a_i^{-1}) \cdot [\widetilde{a}_i]$ to $[\widetilde{w}_j]$. □

The complex $C_*^{cell}(\widetilde{X})$ is a free chain complex over the ring $R = \mathbb{Z}[G']$. Using the lemma, we see that it has the form

$$R^n \xrightarrow{A} R^m \xrightarrow{B} R$$

where

$$A = \psi \begin{bmatrix} d_1 w_1 & d_2 w_2 & \cdots & d_1 w_n \\ d_2 w_1 & d_2 w_2 & \cdots & d_2 w_n \\ \vdots & & \ddots & \vdots \\ d_m w_1 & d_m w_2 & \cdots & d_m w_n \end{bmatrix} \text{ and } B = \begin{bmatrix} \psi(a_1) - 1 & \cdots & \psi(a_m) - 1 \end{bmatrix}.$$

The matrix A is called the *Alexander matrix*.

We now specialize to the case where $G' = \mathbb{Z}$. If our presentation of G comes from a handle decomposition of the knot complement M_K, then $n = m - 1$, and $\widetilde{X}$ is homotopy equivalent to $\overline{M}_K$. In this case, we let A_i be Alexander matrix with its ith row deleted, so A_i is a square matrix.

Proposition 2.4.3. $\det A_i \sim \left(\dfrac{\psi(a_i) - 1}{t - 1} \right) \Delta_K(t)$.

Proof. Let $b_i = \psi(a_i) - 1$ be the ith entry of B, and write $b = \gcd\{b_1, \dots b_{n+1}\}$. Since $\psi(a_i)$'s generate $\mathbb{Z} = \langle t \rangle$, $b = t - 1$. If $\pi_i : R^{n+1} \to R^n$ is the projection which forgets the ith coordinate, then

$$[R^n : \pi_i(\ker B)] \sim \frac{b_i}{b} \sim \frac{\psi(a_i) - 1}{t - 1}.$$

By definition, $\Delta_K(t) \sim \operatorname{ord} H_1(\widetilde{X}) \sim [\ker B : \operatorname{im} A]$. If $b_i \neq 0$, then π_i maps $\ker B$ injectively to R^n, so $[\ker B : \operatorname{im} A] = [\pi_i(\ker B) : \pi_i(\operatorname{im} A)]$. We see that

$$\begin{aligned}\det A_i &\sim [R^n : \pi_i(\operatorname{im} A)] \\ &\sim [\pi_i(\ker B) : \pi_i(\operatorname{im} A)][R^n : \pi_i(\ker B)] \\ &\sim \Delta_K(t)\frac{\psi(a_i)-1}{t-1}.\end{aligned}$$

If $b_i = 0$, let B_i be the matrix obtained by deleting the ith entry from B. We know that $d^2 = 0$ in the cellular chain complex, so $B_i A_i = BA = 0$, which implies that the columns of A_i are linearly dependent. Hence $\det A_i = 0$. □

Example 2.4.4. We use the presentation $\pi_1(M_T) = \langle a, b, c \mid ac^{-1}b^{-1}c, cb^{-1}a^{-1}b\rangle$ from Example 2.1.3, and find that the Alexander matrix is

$$A = \begin{bmatrix} 1 & -t^{-1} \\ -t^{-1} & -1+t^{-1} \\ -1+t^{-1} & 1 \end{bmatrix}.$$

$\psi(a) = \psi(b) = \psi(c) = t$, so we expect $\det A_i \sim 1 - t + t^2$ for each 2×2 minor A_i, as is indeed the case.

The proposition implies that $\Delta_K(t) \sim \gcd \det A_i$. More generally, suppose that $\langle a_1, \dots, a_m \mid w_1, \dots, w_n\rangle$ is a presentation of a group G and $\psi : G \to \mathbb{Z}$ is a surjective homomorphism. Then we can form an Alexander matrix A_ψ exactly as we did above.

Theorem 2.4.5. *Suppose $\psi : G \to \mathbb{Z}$ is as above. Then*

$$\Delta_\psi(G) := \gcd \{\det A'_\psi \mid A'_\psi \text{ is a } m-1 \times m-1 \text{ minor of } A_\psi\} \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$$

is an invariant of the pair (G, ψ) which is well-defined up to multiplication by units in $\mathbb{Z}[t^{\pm 1}]$.

Sketch of proof: see [17]. A Tietze move is one of the following operations on a group presentation:

1) Duplicating a relation.
2) Multiplying one relation by another.
3) Conjugating a relation by a generator.
4) Adding a new generator a' together with the relation $a' = 1$.

Any two presentations of G are related by a sequence of Tietze moves, and one checks directly that the gcd does not change under each of the moves. □

The theorem implies that $\Delta_K(t)$ is both well-defined up to multiplication by $\pm t^k$ ($k \in \mathbb{Z}$) and an invariant of $\pi_1(M_K)$.

2.5. Fibred knots We say that a 3-manifold Y *fibres over* S^1 if there is a submersion $f : Y \to S^1$. If this is the case, all of the fibres $f^{-1}(p)$ are diffeomorphic to some surface Σ, there is a diffeomorphism $\varphi : \Sigma \to \Sigma$ called the *monodromy*, and $Y \simeq \Sigma \times [0,1]/\sim$, where $(x,1) \sim (\varphi(x), 0)$.

Suppose Y is connected, and let θ be a generator of $H^1(S^1)$. We leave it as an exercise to show that the fibre Σ is connected if only if $f^*(\theta)$ is a primitive element of $H^1(Y)$. If $f^*(\theta)$ is divisible by n, then there is a lift $\widetilde{f}$ as shown

$$\begin{array}{ccc} & & S^1 \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow {\scriptstyle z\mapsto z^n} \\ Y & \xrightarrow{f} & S^1 \end{array}$$

and $\widetilde{f}$ is a submersion with connected fibre. From now on, we will only consider fibrations of this form.

We say that a knot K is *fibred* if M_K fibres over S^1.

Example 2.5.1. Identify S^3 with the set $\{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\}$. If p and q are relatively prime, the (p,q) *torus knot* is the set

$$T(p,q) = \{(z,w) \in S^3 \,|\, z^p = w^q\}.$$

The map $f : M_{T(p,q)} \to S^1$ given by

$$f(z,w) = \frac{z^p - w^q}{|z^p - w^q|}$$

is a submersion, and $T(p,q)$ is a fibred knot.

If K is fibred, we can write $M_K \simeq \Sigma \times [0,1]/\sim$, where $(\varphi(x),1) \sim (x,0)$. Consider the map $\Phi : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ given by $\Phi(x,t) = (\varphi(x), t+1)$. Φ generates a free, properly discontinous action of $\mathbb{Z}$ on $\Sigma \times \mathbb{R}$. The set $\Sigma \times [0,1]$ is a fundamental domain for the action of Φ, so $(\Sigma \times \mathbb{R})/\langle\Phi\rangle \simeq \Sigma \times [0,1]/\sim \ \simeq M_K$. The quotient map $p : \Sigma \times \mathbb{R} \to M_K$ is a covering map with deck group $\mathbb{Z}$. The corresponding homomorphism $\pi_1(M_K) \to \mathbb{Z}$ must be the abelianization map, since since any such map factors through $H_1(M_K) \simeq \mathbb{Z}$. Hence p is the infinite cyclic cover. To summarize, we have proved

Proposition 2.5.2. *If M_K fibres over S^1 with monodromy $\varphi : \Sigma \to \Sigma$, then $\overline{M}_K \simeq \Sigma \times \mathbb{R}$. The action of the deck group is generated by the map $(x,t) \to (\varphi(x), t+1)$.*

Corollary 2.5.3. *If M_K fibres over S^1 with monodromy $\varphi : \Sigma \to \Sigma$, then*

$$\Delta_K(t) \sim \det(tI - \varphi_*)$$

where $\varphi_* : H_1(\Sigma) \to H_1(\Sigma)$ is the homomorphism induced by the monodromy.

Proof. For the isomorphism $H_1(\overline{M}_K) \simeq H_1(\Sigma)$, the map $\Phi_* : H_1(\overline{M}_K) \to H_1(\overline{M}_K)$ is given by $\varphi_* : H_1(\Sigma) \to H_1(\Sigma)$. Hence if $e_1, \dots e_{2g}$ is a basis for $H_1(\Sigma)$ over $\mathbb{Z}$, the $\mathbb{Z}[t^{\pm 1}]$-module $H_1(\overline{M}_K)$ will be generated by the e_i, with relations $te_i = \varphi_*(e_i)$. In other words, $H_1(\overline{M}_K)$ has a square presentation matrix of the form $tI - \varphi_*$. □

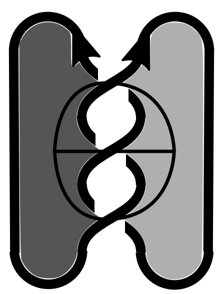
Corollary 2.5.4. *If K is a fibred knot, then $\Delta_K(t)$ is a monic polynomial of degree $2g$, where g is the genus of the fibre.*

The condition of the corollary provides an effective (but not perfect) obstruction to a knot being fibred: a knot of 10 or fewer crossings, is fibred if and only if its Alexander polynomial is monic. More generally, alternating knots are fibred if and only if they have monic Alexander polynomial [57].

2.6. The Seifert genus

Definition 2.6.1. Let K be a knot in S^3. A *Seifert surface* for K is an embedded, orientable, connected surface $\Sigma \hookrightarrow S^3$ with $\partial\Sigma = K$.

Equivalently, we may think of a Seifert surface as being a connected orientable surface Σ with one boundary component that is properly embedded in $S^3 - \nu(K)$ and whose homology class generates $H_2(S^3 - \nu(K), \partial(S^3 - \nu(K)))$. (There are many ways to check the latter condition. Perhaps the easiest is to check that $\partial\Sigma$ represents a nonzero class in $H_1(\partial(S^3 - \nu(K)))$.)



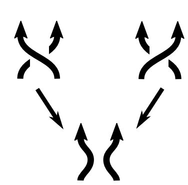
The figure above exhibits a Seifert surface for the trefoil knot T. Note that the surface (being orientable) has two sides, light and dark grey. It is perhaps easiest to visualize the surface as consisting of two large disks of cloth (0-handles) one light side up, the other dark, to which we have attached 3 small strips of cloth (1-handles), one for each crossing. We give each strip half a twist before gluing its ends to the light and dark disks. The surface deformation retracts to the graph shown in black, which has Euler characteristic 2-3 = -1. Hence the surface is homeomorphic to a once-punctured torus.

Lemma 2.6.2. *Any $K \subset S^3$ has a Seifert surface.*

We give two proofs, one constructive, and the other more abstract. Both are worth knowing.

Proof 1. (Seifert's algorithm)

Step 1: Orient K, and replace each crossing with its *oriented resolution*, as shown in the figure to the right. The resulting diagram is a collection of oriented circles, known as *Seifert circles*. Each circle C in the diagram has a *nesting height* $h(C)$ which is equal to the number of other circles one must cross to get from C to the point at infinity in the diagram.



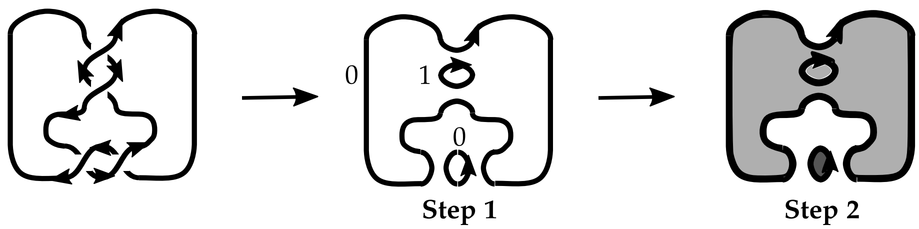
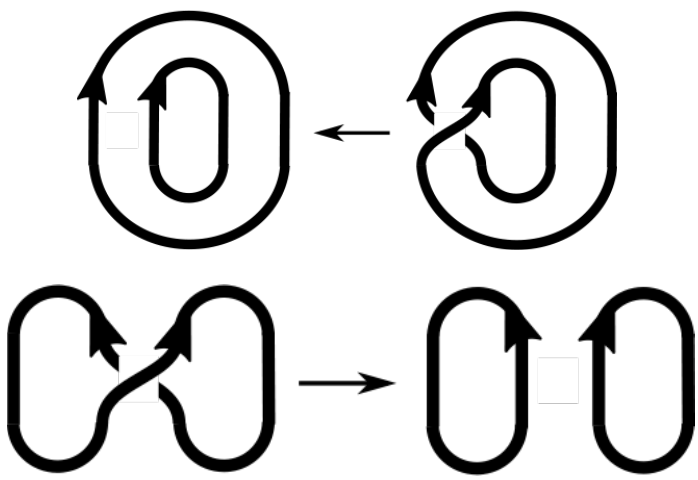
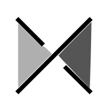


Figure 2.6.3. Seifert's algorithm applied to the figure-8 knot. The numbers in the middle figure indicate the nesting heights of the three Seifert circles. On the right, there are two disks with the light side up: a large one lying in the plane of the paper, and a small one above it.

Step 2: Fill in each circle C with a disk parallel to the xy plane at height $z = h(C)$. If C is oriented clockwise, the dark side of the disk faces up; if it is counterclockwise, the light side. Suppose c is a crossing of our original diagram D. Resolve every crossing other than c and consider the component of the resulting diagram which contains c. It must be in one of the two configurations below.



When we resolve c, we get either a pair of nested disks with the same color facing up, or a pair of unnested disks with opposite colors.



Step 3: Attach a 1-handle for each crossing, as shown in the figure on the right. For visualization purposes, it's helpful to think of the one-handle as a band of cloth with light and dark sides. We give the handle a half twist, and attach it to the disks on either side of the crossing. In each of the two cases considered above, we can arrange that both feet of the one-handle are compatible with the colors of the disks they attach to. Hence the result is an oriented surface. □

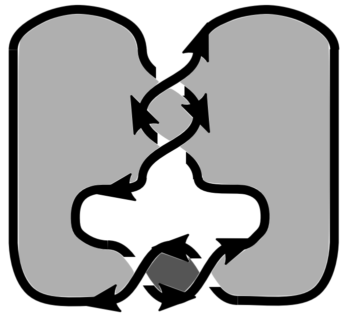


Figure 2.6.4. Seifert surface for the figure-8 knot. The surface in the top half of the figure consists of a large disk (light side up) lying in the plane of the paper together with a twisted band coming out of the paper towards the viewer.

Note that the resulting surface will have $\chi = n - c$, where n is the number of Seifert circles, and c is the number of crossings.

Proof 2. Let $\alpha \in \Omega^1(S^3 - \nu(K))$ be a closed 1-form generating $H^1(S^3 - \nu(K); \mathbb{R})$, and let β be generator of $H_1(S^3 - \nu(K), \mathbb{Z}) \simeq \mathbb{Z}$. After scaling α, we may assume $\langle[\alpha], \beta\rangle = 1$. Fix a basepoint $* \in S^3 - \nu(K)$. There is a well-defined smooth map $f : S^3 - K \to S^1$ given by $f(x) = \int_{\gamma_x} \alpha \mod 1$, where γ_x is any path from $*$ to x, and $f^*(d\theta) = [\alpha]$. If $p \in S^1$ is a regular value of f, then $f^{-1}(p)$ is a submanifold of $S^3 - \nu(K)$ which is Poincare dual to $[\alpha]$. Hence $f^{-1}(p)$ is a Seifert surface for K. □

If Σ is a Seifert surface for K, we can find a Seifert surface of higher genus by taking the connected sum of Σ with a closed surface embedded in S^3. On the other hand, it may not be possible to reduce the genus of Σ. For example, we have

Lemma 2.6.5. *If $K \subset S^3$ bounds an embedded disk, then K is unknotted.*

Proof. Suppose $\Phi : D^2 \to S^3$ is an embedded disk with $\Phi(S^1) = K$. For $t \in (0, 1]$, we define $K_t : S^1 \to S^3$ by $K(\theta) = \Phi(t\theta)$, so Φ is an isotopy between K_t and K_1. Let $T \subset \mathbb{R}^3$ be the tangent plane to $\Phi(D^2)$ at $\Phi(0)$, and let $\pi : \mathbb{R}^3 \to T$ be orthogonal projection. If D^2_ε is the disk of radius ε, the map $\pi \circ \Phi : D^2_\varepsilon \to T$ will be an embedding for small ε. It follows that K_ε is isotopic to the unknot. □

Definition 2.6.6. If K is a knot in S^3, its *Seifert genus* is

$$g(K) = \min\{g(\Sigma) \,|\, \Sigma \text{ is a Seifert surface for } K\}.$$

For example, the trefoil and figure-8 knot both have Seifert surfaces of genus 1. Neither is the unknot, so both knots have genus 1.

One of the most important properties of the Alexander polynomial is the fact that it gives a lower bound on the Seifert genus.

Theorem 2.6.7. *(Seifert)* $2g(K) \geqslant \deg \Delta_K(t)$.

Proof. Suppose $\Sigma \subset M_K$ is a Seifert surface with tubular neighborhood $\nu(\Sigma)$, and let $Y = M_K - \nu(\Sigma)$. Y and Σ are orientable, so $\nu(\Sigma) = \Sigma \times [-1, 1]$. We have maps $\iota^\pm : \Sigma \to \partial Y$ given by the inclusions of $\Sigma \times \{\pm 1\}$, and $M_K \simeq Y/\sim$, where $\iota^-(x) \sim \iota^+(x)$.

Lemma 2.6.8. $H_1(Y) \simeq \mathbb{Z}^{2g}$, *where* $g = g(\Sigma)$.

Proof. We have

$$\begin{aligned} & H^*(S^3, \Sigma) \\ \simeq & H^*(S^3, \nu(\Sigma)) \\ \simeq & H^*(Y, \partial Y) \quad \text{by excision} \\ \simeq & H_{3-*}(Y) \quad \text{by Poincaré duality.} \end{aligned}$$

The long exact sequence of the pair (S^3, Σ) gives

$$0 = H^1(S^3) \to H^1(\Sigma) \to H^2(S^3, \Sigma) \to H^2(S^3) = 0$$

So $H_1(Y) \simeq H^2(S^3, \Sigma) \simeq H^1(\Sigma) \simeq \mathbb{Z}^{2g}$. □

Returning now to the proof of Theorem 2.6.7, let $Z = (Y \times \mathbb{Z})/\sim$, where $(\iota^+(x), t) \sim (\iota^-(x), t+1)$. The map $p : Z \to M_K$ given by $p(y, t) = y$ is a covering map with deck group $\mathbb{Z}$ (generated by the map $(x, t) \mapsto (x, t+1)$), so $Z \simeq \overline{M}_K$. Finally, we have $H_1(Z) = H_1(Y) \otimes \mathbb{Z}[t^{\pm 1}]/\sim$, where the relations $\sim$ are given by $t \cdot \iota^-_*(e) = \iota^+_*(e)$ for $e \in H_1(\Sigma)$. In other words, $H_1(Z) \simeq \operatorname{coker} B$, where $B : R^{2g} \to R^{2g}$ is given by $B(e) = t\iota^-_*(e) - \iota^+_*(e)$. The entries of the $2g \times 2g$ matrix B are linear polynomials in t, so $\operatorname{ord} H_1(Z) = \det A$ is a polynomial in t with degree $\leqslant 2g$. □

Corollary 2.6.9. *If* K *is a fibred knot with fibre* Σ, *then* $g(K) = g(\Sigma)$.

Proof. If Σ is a fibre surface for M_K, $[\Sigma]$ generates $H_2(M_K, \partial M_K)$ (exercise). Hence Σ is a Seifert surface for K. By Corollary 2.5.4, $2g(\Sigma) = \deg \Delta_K(t) \leqslant 2g(K)$, so $g(\Sigma) = g(K)$. □

As with Corollary 2.5.4, the bound of Theorem 2.6.7 is sharp for many knots, including all alternating knots [16,56] and all knots of 10 crossings or fewer. However there are many nontrivial knots K with $\Delta_K(t) = 1$. (Perhaps the simplest example is the pretzel knot $P(-3, 5, 7)$ from Figure 1.1.3.) Any such knot has $g(K) \geqslant 1$, so the inequality of Theorem 2.6.7 is not sharp.

2.7. The Seifert Matrix The maps $\iota^\pm_*$ can be described much more concretely. Let $\alpha : H_1(Y) \to H^1(\Sigma)$ be the isomorphism of Lemma 2.6.8.

Lemma 2.7.1. *If* $y \in H_1(\Sigma)$, *then* $\langle \alpha(x), y \rangle = \operatorname{lk}(x, \iota_*(y))$, *where* $\iota : \Sigma \to S^3$ *is the inclusion.*

Here $\operatorname{lk}(a, b)$ is the linking number of the two-component link $a \amalg b$, as discussed in the exercises to section 1.

Proof. The Poincaré duality isomorphism $\mathrm{PD} : H_1(Y) \to H^2(Y, \partial Y)$ satisfies

$$\langle \mathrm{PD}(x), S\rangle = x \cdot S$$

for $S \in H_2(Y, \partial Y)$ and the pair of boundary maps $\delta : H^1(\Sigma) \to H^2(S^3, \Sigma)$ and $\partial : H_2(S^3, \Sigma) \to H_1(\Sigma)$ are dual to each other, so

$$\langle \alpha(x), y\rangle = \langle \delta^{-1}(\mathrm{PD}(x)), y\rangle = \langle \mathrm{PD}(x), \partial^{-1}(y)\rangle = x \cdot \partial^{-1}(y)$$

The class $\partial^{-1}(y)$ is represented by an embedded surface bounding $\iota_*(y)$, so we have $x \cdot \partial^{-1}(y) = \mathrm{lk}(x, \iota_*(y))$. □

Choose embedded oriented curves $x_1, \ldots, x_{2g} \subset \Sigma$ which form a basis of $H_1(\Sigma)$, and let $x^1, \ldots x^{2g}$ be the dual basis of $H^1(\Sigma)$. We define $\alpha^\pm : H_1(\Sigma) \to H^1(\Sigma)$ by $\alpha^\pm(x) = \alpha(\iota_*^\pm(x))$.

The lemma says that with respect to the bases $\{x_1, \ldots, x_{2g}\}$ and $\{x^1, \ldots, x^{2g}\}$, the map α^+ is represented by the $2g \times 2g$ matrix A^+ with entries

$$a_{ij}^+ = \mathrm{lk}(\iota_*^+(x_j), \iota_*(x_i)),$$

while α^- is represented by the $2g \times 2g$ matrix A^- with entries

$$a_{ij}^- = \mathrm{lk}(\iota_*^-(x_j), \iota_*(x_i)).$$

Equivalently, using the bases $\{x_1, \ldots, x_{2g}\}$ for $H_1(\Sigma)$ and $\{\alpha^{-1}(x^1), \ldots, \alpha^{-1}(x^{2g})\}$ for $H_1(Y)$, the maps $\iota_*^\pm : H_1(\Sigma) \to H_1(Y)$ are given by the matrices $A^\pm$. Comparing with end of the proof of Theorem 2.6.7, we see that

$$\Delta_K(t) \sim \det B = \det(tA^- - A^+).$$

The link $\iota^-(x_j) \cup \iota(x_i)$ is isotopic to $\iota(x_j) \cup \iota^+(x_i)$, so $a_{ij}^- = a_{ji}^+$, or equivalently, $A^- = (A^+)^T$.

Definition 2.7.2. If Σ is a Seifert surface for K and $x_1, \ldots, x_{2g} \subset \Sigma$ is a basis of $H_1(\Sigma)$, the associated *Seifert matrix* $A := A^+$ is given by

$$A = [a_{ij}] = [\mathrm{lk}(\iota_*^+(x_j), \iota_*(x_i))].$$

To sum up, we have proved

Proposition 2.7.3. $\Delta_K(t) \sim \det(A - tA^T)$, *where* A *is a Seifert matrix of* K.

The Seifert matrix lets us pick out a canonical representative for the Alexander polynomial as an element of $\mathbb{Z}[t^{\pm 1}]$.

Definition 2.7.4. The *symmetrized Alexander polynomial* of a knot $K \subset S^3$ is defined to be

$$\Delta_K(t) = \det(t^{-1/2}A - t^{1/2}A^T),$$

where A is a Seifert matrix for K.

Proposition 2.7.5. *The symmetrized Alexander polynomial satisfies the following properties:*

1) *(Integrality)* $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$.

2) *(Symmetry)* $\Delta_K(t^{-1}) = \Delta_K(t)$.
3) *(Normalization)* $\Delta_K(1) = 1$.

Proof of Proposition 2.7.5. For item 1), note that A is a $2g \times 2g$ matrix, so

$$\det(t^{-1/2}A - t^{1/2}A^T) = t^{-g}\det(A - tA^T).$$

For 2) note that the expression $\det(t^{-1/2}A - t^{1/2}A^T)$ is invariant under the involution of $\mathbb{Z}[t^{\pm 1/2}]$ which sends $t^{1/2}$ to $-t^{-1/2}$. The restriction of this involution to $\mathbb{Z}[t^{\pm 1}]$ sends t to t^{-1}.

Finally, for property 3), note that $\Delta_K(1) = \det(A - A^T) = \det(C)$, where

$$c_{ij} = \mathrm{lk}(\iota_*^+(x_j), \iota_*(x_i)) - \mathrm{lk}(\iota_*^-(x_j), \iota_*(x_i)) = X_j \cdot x_i,$$

where X_j is the annulus $x_j \times [-1, 1] \subset \Sigma \times [-1, 1]$. On the other hand, it is easy to see that $X_j \cdot x_i = x_j \cdot x_i$, where the intersection number on the left is in S^3, and the one on the right is taken in Σ. In other words C is the matrix representing the intersection form on Σ with respect to the basis $x_1, \dots x_{2g}$. Hence $\det C = 1$. □

Remark 2.7.6. In order for Definition 2.7.4 to make sense, we must check that it does not depend on the choice of Seifert matrix A. This follows from Proposition 2.7.5. Indeed, if we are given are two Seifert matrices A_1 and A_2 for K, let $p_i(t) = \det(t^{-1/2}A_i - t^{1/2}A_i^T)$. Then $p_1(t)$ and $p_2(t)$ are both normalized and symmetric, and $p_1(t) \sim \Delta_K(t) \sim p_2(t)$. It is easy to see that this implies $p_1 = p_2$.

2.8. Links If $L \subset S^3$ is a link, we have $H_1(S^3 - L) = \mathbb{Z}^{|L|}$, so there are many surjective homomorphisms $\varphi : \pi_1(S^3 - L) \to \mathbb{Z}$. Each such homomorphism gives rise to an infinite cyclic cover X_φ, and we can define a corresponding Alexander polynomial $\Delta_\varphi(L)$ to be the order of $H_1(X_\varphi; \mathbb{Q})$ as a module over $R = \mathbb{Q}[t^{\pm 1}]$. Note that $H_1(X_\varphi; \mathbb{Q})$ need not be a torsion module over R; if it is not, its order $\Delta_\varphi(L)$ is defined to be 0. (With this definition, we still have $\operatorname{ord} M = \det A$ if A is a $n \times n$ presentation matrix for M.)

If L is an oriented link with n components, the oriented meridians $m_1, \dots, m_n$ form a basis for $H_1(S^3 - L)$.

Definition 2.8.1. If L is an oriented link, we define $\Delta(L) = \Delta_\varphi(L)$ where the homomorphism $\varphi : \pi_1(S^3 - L) \to \mathbb{R}$ is given by $\varphi(m_i) = 1$ for all i. Note that if L^r denotes L with the orientation on each component reversed, the homomorphisms corresponding to L and L^r have the same kernel, so $\Delta(L) = \Delta(L^r)$.

Similarly, if L is an oriented link, we define a Seifert surface for L to be an embedded oriented surface whose oriented boundary is L. With this definition, the results of section 2.6 carry over more or less unchanged. In particular, if Σ is a Seifert surface for L, the degree of the (unsymmetrized) Alexander polynomial $\Delta(L)$ gives a lower bound on the rank of $H_1(\Sigma)$.

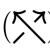
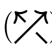
Note that changing the orientation of some (but not all) components of L will usually have a drastic effect on the Alexander polynomial and Seifert genus; see Figure 2.8.2 for an example.

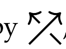
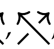
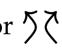
$\Delta(L) = t^{3/2} - t^{1/2} + t^{-1/2} - t^{-3/2}$ $\qquad$ $\Delta(L) = 2t^{1/2} - 2t^{-1/2}$

Figure 2.8.2. Minimal genus Seifert surfaces for two different orientations on the torus link $T(2,4)$. The corresponding Alexander polynomials are shown.

Finally, we remark that if L is an oriented link, and Σ is a Seifert surface for L, then the definition of the Seifert matrix given in Definition 2.7.2 still makes sense. As before, we define the symmetrized Alexander polynomial $\Delta_L(t)$ to be $\det(t^{-1/2}A - t^{1/2}A^T)$, where A is the Seifert matrix.

We can use the Seifert matrix to prove that the Alexander polynomial satisfies a skein relation similar to the oriented skein relation satisfied by the Jones polynomial.

Proposition 2.8.3. *(Skein Relation)* Δ () $- \Delta$ () $= (t^{1/2} - t^{-1/2})$ () .

Proof. Let $D_\pm$ and D_0 be the three planar diagrams represented by , , and , and let Σ_0 be the Seifert surface for  obtained by applying Seifert's algorithm to D_0. The surfaces $\Sigma_\pm$ that we get by applying Seifert's algorithm to $D_\pm$ are the result of attaching one more 1-handle to Σ_0.

If we choose a basis $x_1, \ldots, x_n$ for $H_1(\Sigma_0)$, we get bases for $H_1(\Sigma_\pm)$ by appending a single curve $x_\pm$, which runs from a point p in Σ_0, over the new 1-handle, and back to p. Clearly, these satisfy $\mathrm{lk}(i_*^+(x_+), i_*(x_i)) = \mathrm{lk}(i_*^+(x_-), i_*(x_i))$. Similarly, the diagrams for the two-component links $i_*^+(x_+) \amalg i_*(x_+)$ and $i_*^+(x_-) \amalg i_*(x_-)$ will be the same except in a neighborhood of the new crossing, where the first diagram will have a positive crossing and the second diagram will have a negative one.

It follows that if A_0 is a Seifert matrix for Σ_0, the Seifert matrix $A_\pm$ for $\Sigma_\pm$ will have the form

$$A_+ = \begin{bmatrix} A_0 & \mathbf{x} \\ \mathbf{y} & k \end{bmatrix} \qquad A_- = \begin{bmatrix} A_0 & \mathbf{x} \\ \mathbf{y} & k-1 \end{bmatrix}.$$

If $B_0 = t^{-1/2}A_0 - t^{1/2}A_0^T$ and similarly for $B_\pm$, we have

$$B_+ = \begin{bmatrix} B_0 & \mathbf{z} \\ \mathbf{w} & (t^{-1/2} - t^{1/2})k \end{bmatrix} \qquad B_- = \begin{bmatrix} B_0 & \mathbf{z} \\ \mathbf{w} & (t^{-1/2} - t^{1/2})(k-1) \end{bmatrix}.$$

Expanding both determinants along the lower row, we see that

$$\det B_+ - \det B_- = (t^{-1/2} - t^{1/2}) \det B_0$$

which gives the skein relation stated above. □

2.9. Connections and Further Reading The Alexander polynomial of a knot is a special case of a more general invariant of 3-manifolds the multi-variable Alexander polynomial defined as follows. If Y is a 3-manifold with toroidal boundary, we can consider the covering space $\widetilde{Y}$ corresponding to the surjective map $\pi_1(Y) \to H_1(Y) \to H_1(Y)/\mathrm{Tors}$ where $\mathrm{Tors} \subset H_1(Y)$ is the torsion subgroup. If $H_1(Y)$ has rank k, $H_1(\widetilde{Y})$ is a module over $\mathbb{Z}[\mathbb{Z}^k]$, which is the ring of Laurent polynomials in k variables. This ring is not a PID, but it is a UFD, so we can use a definition analogous to that of Theorem 2.4.5 to get an invariant $\Delta(Y) \in \mathbb{Z}[H_1(Y)/\mathrm{Tors}]$. In the case where $Y = S^3 - \nu(L)$, the single-variable Alexander polynomial can be obtained from $\Delta(Y)$ by specialization and multiplication by an appropriate factor. There are useful analogs of Theorem 2.6.7 and Corollary 2.5.4 which relate the multivariable Alexander polynomial to the Thurston norm. McMullen's paper [52] is a good reference.

Closely related to the multivariable Alexander polynomial is the notion of the Alexander polynomial as a *torsion*, which was developed in a beautiful paper by Milnor [54]. A key property of the torsion is that it satisfies a product formula, which relates the torsion of two manifolds glued together along a torus to the torsion of the individual pieces. This is especially useful for understanding the effect of Dehn filling and satellite operations on the Alexander polynomial. Turaev's book [79] provides a nice introduction to this subject.

Exercises

1. Find a presentation of $\pi_1(M_K)$, where K is the figure-8 knot. Show that it can be reduced to a 2-generator, 1-relator presentation. Compute the Alexander polynomial using Fox calculus and using the skein relation.
2. Show that $S^3 - T(p,q)$ can be written as the union of two solid tori glued together along an annulus. Deduce that $\pi_1(M_{T(p,q)}) = \langle a, b \mid a^p b^q = 1 \rangle$. Compute $\Delta(T(p,q))$. What is $g(T(p,q))$?
3. If L is a split link (*i.e* a disjoint union of two links), show that $\Delta(L) = 0$ by a) showing that $H_1(\widetilde{M_L})$ is not a torsion module and b) using the skein relation.
4. If K is a knot in S^3, use the fact that $H_1(M_K) = \mathbb{Z}$ to show that $\Delta_K(1) = 1$.
5. Use the skein relation to show that $\Delta_K(-1) = V_K(-1)$. The quantity $|\Delta_K(-1)|$ appears in many contexts, and is known as the *determinant* of K.
6. Let L be the $(2, 2n)$ torus link, oriented so that it is the boundary of an embedded annulus, as in the right-hand link in Figure 2.8.2. Compute $\Delta(L)$ both by using the skein relation and by finding a Seifert matrix.
7. Let $P(p,q,r)$ be the (p,q,r) pretzel knot, as in Figure 1.1.3, where p, q and r are all odd. Show that K bounds a genus 1 Seifert surface. Find the associated Seifert matrix and compute $\Delta(K)$. Use this to show that $\Delta(P(-3,5,7)) = 1$. Compare this with Exercise 7, section 1), which shows that $P(-3,5,7)$ is not the unknot.

8. Suppose that H_g is a 3-dimensional handlebody obtained by starting with a 0-handle and attaching g 1-handles $D^1 \times D^2$ in an oriented fashion. Consider the compressing disks $0 \times D^2$ inside the one-handles, and let $\alpha_1, \dots, \alpha_g \subset \Sigma_g$ be their boundaries. If β is a curve in Σ_g, let w be its image in $\pi_1(H_g) = \langle a_1, \dots, a_g \rangle$. Explain why there is a bijection between $\beta \cap \alpha_i$ and occurrences of $a_i^{\pm 1}$ in the word w. Now suppose that $\beta_1, \dots, \beta_g$ are g distinct curves in Σ_g and consider the set of unordered g-tuples of points $\{p_1, \dots, p_g\}$, where $p_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in S_g$.

   Show that the elements of this set are in bijection with monomials in $\det d_i w_j$, where we expand everything out without cancelling any monomials with opposite signs.
9. We say K is a g-bridge knot if K has a diagram with k maxima for the z-coordinate. If this is the case, show that $S^3 - \nu(K)$ can be decomposed as a handlebody of genus K with $k-1$ 2-handles attached. (Hint: we can assume the maxima all occur at $z = 1$ and the minima at $z = -1$. Consider the intersection of K with the half-spaces $z \geqslant 0$ and $z \leqslant 0$.) Can you draw the resulting handle decomposition for the complement of the trefoil?

3. Khovanov Homology

In this lecture, we transition from studying polynomial invariants of knots to their categorifications. Khovanov homology is an invariant of oriented knots and links in S^3. It is a homological generalization of the Jones polynomial in the following sense: if $L \subset S^3$ is an oriented link, its Khovanov homology is a bigraded abelian group $\mathrm{Kh}^{i,j}(L)$, whose graded Euler characteristic

$$\chi(\mathrm{Kh}(L)) := \sum_{i,j} (-1)^i q^j \dim \mathrm{Kh}^{i,j}(L) = \overline{V}(L).$$

To define $\mathrm{Kh}(L)$, we first represent L by a planar diagram D. To such a diagram, Khovanov assigns a bigraded chain complex $\mathrm{CKh}(D)$; $\mathrm{Kh}(L)$ is its homology. Khovanov showed that if D_1 and D_2 are two planar diagrams representing the same link L, $\mathrm{CKh}(D_1)$ and $\mathrm{CKh}(D_2)$ are chain homotopy equivalent, hence have the same homology.

The definition of $\mathrm{CKh}(D)$ is neatly encapsulated in the following

Slogan ([3]). $\mathrm{CKh}(D)$ is obtained by applying a certain $1+1$ dimensional TQFT $\mathcal{A}$ to the cube of resolutions of D.

whose meaning we now explain.

3.1. Cube of resolutions If c is a crossing of the diagram D, we may turn the paper (or our head) until the crossing appears as shown in Figure 3.1.1. As illustrated in the figure, the crossing c can be resolved in two ways, which we call the 0 and 1-resolutions.

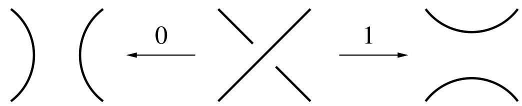


Figure 3.1.1. The 0 and 1 resolutions of a crossing

Vertices: If D has n crossings, there are 2^n ways to resolve all n of them, which (after ordering the crossings) are naturally in bijection with the vertices of the cube $[0,1]^n$. If v is a vertex of the cube, we write D_v for the planar diagram of the corresponding resolution. D_v has no crossings, so it is a collection of embedded circles in the plane. We will (mostly) ignore the embedding, and view D_v as a 1-dimensional manifold.

The figure below illustrates this process for the Hopf link.

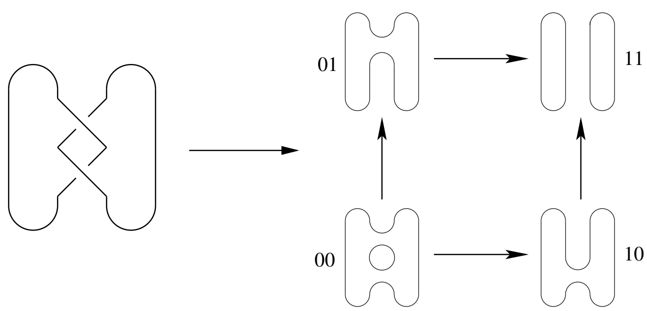


Figure 3.1.2. The cube of resolutions of the Hopf link

Edges: Along each edge e of the cube, one coordinate varies from 0 to 1, while all the other coordinates are fixed. We orient e to point from the vertex v_0 where the variable coordinate is 0 to the vertex v_1 where the variable component is 1, as shown in Figure 3.1.2, and write $e : v_0 \to v_1$.

To each edge $e : v_0 \to v_1$, we assign a surface S_e with $\partial S_e = D_{v_0} \cup D_{v_1}$. The diagrams D_{v_0} and D_{v_1} are identical away from a neighborhood of a single crossing c, and we define S_e to be the product $D_{v_0} \times I$ away from this neighborhood. Inside this neighborhood, S_e is given by the saddle cobordism shown in Figure 3.1.3.

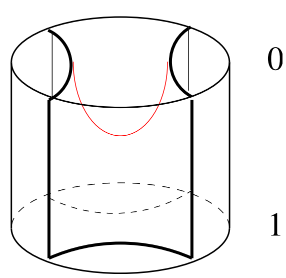


Figure 3.1.3. Cobordism from D_{v_0} to D_{v_1}

Equivalently, S_e is obtained from $D_{v_0} \times I$ by attaching a 1-handle whose core is the red curve shown in the figure.

3.2. The Cobordism Category If Y_1 and Y_2 are compact oriented n-manifolds, we define a *cobordism* W from Y_1 to Y_2 to be a compact oriented $n+1$-manifold with $\partial W = -Y_1 \coprod Y_2$. We write $W : Y_1 \to Y_2$. Two cobordisms $W, W' : Y_1 \to Y_2$ are *equivalent* if there is a homeomorphism $f : W \to W'$ whose restriction to ∂W is the identity. If $W_1 : Y_1 \to Y_2$ and $W_2 : Y_2 \to Y_3$ are cobordisms, their *composition* $W_2 \circ W_1 := W_1 \cup_{Y_2} W_2$ is a cobordism from $Y_1 \to Y_3$, as shown in Figure 3.2.1.

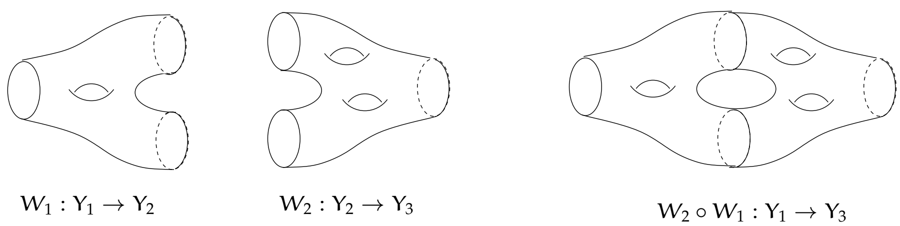


Figure 3.2.1. Composition of Cobordisms

We define the $n+1$ *dimensional cobordism category* to be the category whose objects are compact oriented $n-$manifolds and whose morphisms are equivalence classes of cobordisms between them. If Y is an object in this category, the identity morphism 1_Y is the product $Y \times [0,1]$.

We would like to view the vertices of the cube of resolutions as being decorated by objects of the $1+1$ dimensional cobordism category, and its edges as being decorated by morphisms. To do this, we must orient the objects involved. First note that any circle embedded in $\mathbb{R}^2$ has a standard (counterclockwise) orientation.

Definition 3.2.2. Let D_v be a set of embedded circles in $\mathbb{R}^2$. The *canonical orientation* on D_v is defined by giving the ith circle C_i $(-1)^{n_i}$ times the standard orientation, where n_i is the number of circles separating C_i from infinity in $\mathbb{R}^2$.

This process is illustrated in the figure below.

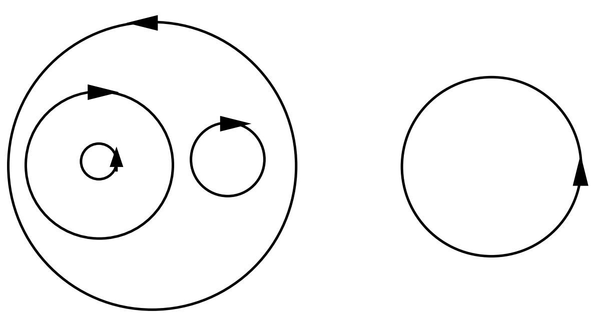


Figure 3.2.3. The canonical orientation on circles embedded in $\mathbb{R}^2$.

Exercise 3.2.4. Let $e : v_0 \to v_1$ be an edge in the cube of resolutions. If we give D_{v_0} and D_{v_1} their canonical orientations, show that S_e is an oriented cobordism from D_{v_0} to D_{v_1}.

Remark 3.2.5. Since S_e is given by a single one-handle attachment and is orientable, it is the disjoint union of a single pair of pants with some cylinders.

Exercise 3.2.6. A knot $K \subset T^2 \times I$ can be represented by a planar diagram in T^2, and we can form the cube of resolutions for K as above. Show by example that the surfaces S_e may be nonorientable in this case.

Faces: Each 2-dimensional face of the cube of resolutions corresponds to a square of morphisms in the cobordism category. This square commutes, since 1-handles can be added in any order without changing the homeomorphism type of the resulting surface. The situation is summarized in the first two columns of the table below.

vertex v	$\Longrightarrow$	1-manifold D_v	$\Longrightarrow$	group $\mathcal{A}(D_v)$
edge $e : v_0 \to v_1$	$\Longrightarrow$	cobordism $S_e : D_{v_0} \to D_{v_1}$	$\Longrightarrow$	linear map $\mathcal{A}(S_e) : \mathcal{A}(D_{v_0}) \to \mathcal{A}(D_{v_1})$
2 dim'l faces		commute		commute

3.3. Applying a TQFT A $1+1$ dimensional topological quantum field theory (or TQFT, for short) is a monoidal functor

$$\mathcal{A} : \begin{Bmatrix} \text{1-manifolds} \\ \text{cobordisms} \end{Bmatrix} \to \begin{Bmatrix} \text{abelian groups} \\ \text{linear maps} \end{Bmatrix}$$

Saying that that $\mathcal{A}$ is *monoidal* means that it behaves well under disjoint unions: if Y and Y′ are 1-manifolds, then $\mathcal{A}(Y \coprod Y') = \mathcal{A}(Y) \otimes \mathcal{A}(Y)$. Similarly if $W : Y_1 \to Y_2$, $W' : Y_1' \to Y_2'$ are morphisms, then $\mathcal{A}(W \coprod W') = \mathcal{A}(W) \otimes \mathcal{A}(W')$.

Applying $\mathcal{A}$ to the cube of resolutions, we get a cube whose vertices are labeled by abelian groups $\mathcal{A}(D_v)$, and whose edges are labeled by linear maps $\mathcal{A}(S_e)$. Since $\mathcal{A}$ is a functor, the two dimensional faces in this cube still commute. The situation is summarized in the last column of the table above.

We can now define the Khovanov complex. As a group

$$\mathrm{CKh}(D) = \bigoplus_v \mathcal{A}(D_v)$$

where the sum runs over all vertices of the cube $[0,1]^n$. For $x \in \mathcal{A}(D_v)$, the differential on $\mathrm{CKh}(D)$ is given by

$$dx = \sum_{e: v \to v'} (-1)^{\sigma(e)} \mathcal{A}(S_e)(x)$$

where σ is a map from the set of edges to $\{0,1\}$, which we insert in order to make $d^2 = 0$. Indeed, we have

Lemma 3.3.1. *If σ is chosen so that each two dimensional face of the cube has an odd number of edges with $\sigma(e) = 1$, then $d^2 = 0$.*

Proof. If v'' is a vertex of the cube obtained by changing two 0 coordinates of v to 1's, then the component of $d^2(x)$ which lies in the summand in $\mathcal{A}(v'')$ is

$$(-1)^{\sigma(e_1)+\sigma(e_2)}\mathcal{A}(S_{e_2}) \circ \mathcal{A}(S_{e_1})(x) + (-1)^{\sigma(e_3)+\sigma(e_4)}\mathcal{A}(S_{e_3}) \circ \mathcal{A}(S_{e_4})(x)$$

where e_1, e_2, e_3 and e_4 are the edges of the two dimensional face containing v and v'', labeled clockwise starting from v. Since two-dimensional faces of the cube commute,

$$\mathcal{A}(S_{e_2}) \circ \mathcal{A}(S_{e_1}) = \mathcal{A}(S_{e_3}) \circ \mathcal{A}(S_{e_4})$$

Thus to ensure that $d^2 = 0$, it suffices to choose σ so that each two-dimensional face of the cube has an odd number of edges with $\sigma(e) = 1$. □

The following exercise shows that it is possible to choose such a σ, and that any two such choices give rise to isomorphic chain complexes.

Exercise 3.3.2. Let $C^* = C^*_{cell}([0, 1]^n; \mathbb{Z}/2)$ be the cellular cochain complex of the cube with $\mathbb{Z}/2$ coefficients.

(1) Viewing σ as an element of C^1, show that the condition of the lemma is satisified if and only if $d\sigma = \tau$, where $\tau \in C^2$ is the cochain which assigns 1 to each two-dimensional face of the cube. Deduce that it is possible to choose σ satisfying the conditions of the lemma.
(2) Suppose $\sigma' \in C^1$ also satisfies $d\sigma' = \tau$, and let $CKh(D)$ and $CKh'(D)$ be the Khovanov complexes defined using σ and σ'. Show that $\sigma - \sigma' = d\rho$ for some $\rho \in C^0$, and use ρ to define an isomorphism between $CKh(D)$ and $CKh'(D)$.

3.4. The TQFT $\mathcal{A}$ Up until this point, the construction we have described works for any TQFT, but the resulting homology depends on the planar diagram D, rather than its underlying link L. To get a chain complex whose homology is a link invariant, we will use a particular TQFT $\mathcal{A}$ for which

$$\mathcal{A}(S^1) = \langle \mathbf{1}, \mathbf{x} \rangle =: V$$

is a free abelian group of rank two. Since $\mathcal{A}$ is a monoidal functor, we must have

$$\mathcal{A}(\coprod^n S^1) = V^{\otimes n}.$$

This completely specifies the functor $\mathcal{A}$ at the level of objects.

If D_v is a closed 1-manifold, we define a *state* of D_v to be a labeling of each component of D_v by either **1** or **x**. The $\mathbb{Z}$-module $\mathcal{A}(D_v)$ has a basis consisting of states of D_v.

More generally, if D is a planar diagram, we define a *state* of D to be a choice of a complete resolution of D, together with a state of the complete resolution. $CKh(D)$ has a basis consisting of states of D.

Example 3.4.1. If D is the diagram of the Hopf link shown in Figure 3.1.2, $\mathsf{CKh}(D)$ has rank 12. A set of generators for the complex is shown in Figure 3.4.2.

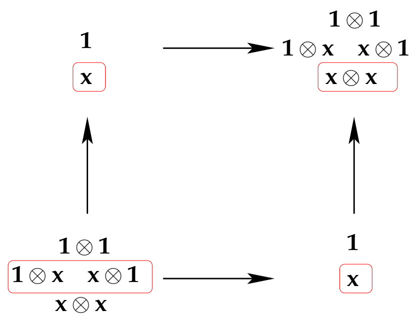


Figure 3.4.2. $\mathsf{CKh}(D)$, where D is the diagram of the Hopf link shown in Figure 3.1.2. The generators in the boxes all have the same q-grading.

Morse theory tells us that any cobordism is the composition of handle attachments. It follows that any morphism in the cobordism category can be built up as a composition of the *elementary cobordisms* shown in Figure 3.4.3 and their disjoint unions with cylinders

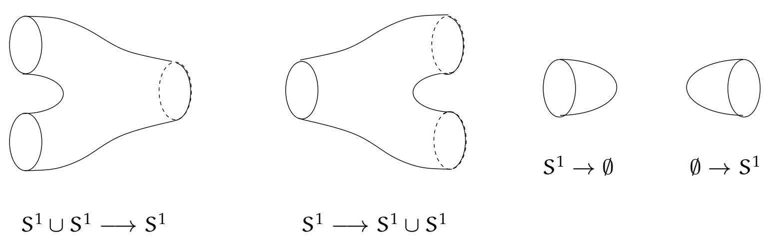


Figure 3.4.3. Elementary cobordisms: merge, split, death, and birth

The functor $\mathcal{A}$ is monoidal, so to understand how it acts on morphisms, it is enough to describe it for the cobordisms shown in the figure. The split and merge morphisms give rise to maps

$$\begin{aligned} m : V \otimes V &\to V \\ \mathbf{1} \otimes \mathbf{1} &\mapsto \mathbf{1} \\ \mathbf{1} \otimes \mathbf{x}, \mathbf{x} \otimes \mathbf{1} &\mapsto \mathbf{x} \\ \mathbf{x} \otimes \mathbf{x} &\mapsto 0 \end{aligned} \qquad \begin{aligned} \Delta : V &\to V \otimes V \\ \mathbf{1} &\mapsto \mathbf{1} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{1} \\ \mathbf{x} &\mapsto \mathbf{x} \otimes \mathbf{x} \end{aligned}$$

while the birth and death morphisms give maps

$$\begin{aligned} \eta : \mathbb{Z} &\to V \\ 1 &\mapsto \mathbf{1} \end{aligned} \qquad \begin{aligned} \varepsilon : V &\to \mathbb{Z} \\ \mathbf{1} &\mapsto 0 \\ \mathbf{x} &\mapsto 1. \end{aligned}$$

We have now given a complete definition of the complex $CKh(D)$. Given enough patience and/or computer memory, it is straightforward to compute the complex and its homology for any planar diagram D.

Exercise 3.4.4. Compute the homology of the complex shown in Figure 3.4.2.

Exercise 3.4.5. Show that $CKh(D)$ and $CKh(\overline{D})$ are dual chain complexes.

Exercise 3.4.6. Suppose that $\mathcal{A}$ is a $1+1$ dimensional TQFT, and let $A = \mathcal{A}(S^1)$. A can be equipped with maps $m : A \otimes A \to A$, $\Delta : A \to A \otimes A$, $1 : \mathbb{Z} \to A$, and $\varepsilon : A \to \mathbb{Z}$ as above. Let $\iota : A \otimes A \to A \otimes A$ be the involution given by $\iota(a \otimes b) = b \otimes a$. Use the fact that $\mathcal{A}$ is a TQFT to show that

(1) $m(a \otimes b) = m(b \otimes a)$ and $m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$
(2) $\iota \circ \Delta = \Delta$ and $(1_A \otimes \Delta) \circ \Delta = (\Delta \otimes 1_A) \circ \Delta$
(3) $m(\mathbf{1} \otimes a) = a$ and $(\varepsilon \otimes 1_A) \circ \Delta = 1_A$
(4) $\Delta \circ m = (m \otimes 1_A) \circ (1_A \otimes \Delta)$.

A set A equipped with maps m (multiplication), Δ (comultiplication), 1 (unit) and ε (counit) satisfying the properties above is called a *Frobenius algebra*. The exercise above shows that any $1+1$ dimensional TQFT determines a Frobenius algebra. Conversely, it can be shown [46] that any Frobenius algebra determines such a TQFT.

At this point, it is quite natural to ask whether it is possible to replace the particular TQFT $\mathcal{A}$ used above with some other TQFT $\mathcal{A}'$. As the following exercise shows, the possible choices of $\mathcal{A}$ are strongly restricted by the condition that the homology should be a link invariant. (In fact, we will see in section 3.8 that the TQFT used here is essentially the only one whose homology gives an interesting link invariant.)

Exercise 3.4.7. Suppose $\mathcal{A}'$ is a $1+1$ dimensional TQFT for which $\mathcal{A}'(S^1)$ is a free abelian group of rank n. If D is a planar diagram, let $CKh_{\mathcal{A}'}$ be the result of applying the construction of section 3.3 with $\mathcal{A}'$ in place of $\mathcal{A}$. Let D be the zero-crossing diagram of the unknot, and let D' be a one-crossing diagram of the unknot. Show that if $CKh_{\mathcal{A}'}(D)$ and $CKh_{\mathcal{A}'}(D')$ have isomorphic homology, then $n = 2$.

3.5. Gradings $CKh(D)$ can be equipped with a natural bigrading:

$$CKh(D) = \bigoplus_{i,j} CKh^{i,j}(D)$$

such that $d : CKh^{i,j} \to CKh^{i+1,j}$. The first grading is the *(co)homological* grading on $CKh(D)$. If v is a vertex of $[0,1]^n$, we write $|v|$ for the sum of the coefficients of v. An element of $\mathcal{A}(D_v)$ has (co)homological grading $|v|$. It is clear from the definition that d raises the cohomological grading by 1. To define the second grading, which we call the q-grading, we first define a grading $\widetilde{q}$ on V by

$$\widetilde{q}(\mathbf{1}) = 1 \qquad \widetilde{q}(\mathbf{x}) = -1$$

and extend it to $V^{\otimes n}$ by the relation $\widetilde{q}(a \otimes b) = \widetilde{q}(a) + \widetilde{q}(b)$. With respect to this grading, $\mathcal{A}$ is a *graded TQFT*, in the sense that

$$\widetilde{q}(\mathcal{A}(S)(x)) = \chi(S) + \widetilde{q}(x)$$

for any cobordism S. The reader can easily check this for the maps induced by the four elementary cobordisms above (note that any element of the ground ring $\mathbb{Z}$ has $\widetilde{q}$-grading 0). Any cobordism S is a composition of elementary cobordisms, so the general case follows from the fact that $\chi(S_1 \circ S_2) = \chi(S_1) + \chi(S_2)$.

Since each cobordism S_e is the union of a pair of pants with some cylinders, we see that if $x \in \mathcal{A}(D_v)$, then $\widetilde{q}(dx) = \widetilde{q}(x) - 1$. For $x \in \mathcal{A}(D_v)$, we define

$$q(x) = \widetilde{q}(x) + |v|$$

so that $q(dx) = q(x)$.

Remark 3.5.1. As the above discussion shows, $\widetilde{q}$ is also a natural choice of homological grading on $CKh(D)$; it appears as the homological grading in Seidel and Smith's *symplectic Khovanov homology* [73].

Since d preserves the q grading, $CKh(D)$ decomposes as a direct sum of chain complexes:

$$CKh(D) = \bigoplus_j CKh^{*,j}(D).$$

We can combine the Euler characteristics of the summands together to form a generating function

$$\chi(Kh(D)) := \sum_{i,j} (-1)^i q^j \operatorname{rk} CKh^{i,j}(D)$$

which is called the *graded Euler characteristic* of $Kh(D)$.

If we further define the the *bigraded Poincaré polynomial* of $Kh(D)$ to be

$$P(Kh(D)) := \sum_{i,j} t^i q^j \operatorname{rk} Kh^{i,j}(D),$$

then the usual argument shows that $\chi(Kh(D)) = P(Kh(D))|_{t=-1}$.

3.6. Invariance Let D be a planar diagram of L. To pin down the exact normalization of the Jones polynomial $V(L)$, we needed to fix an orientation on L and shift $\langle D \rangle$ by some power of q depending on the number of positive and negative crossings in D. The situation for Kh is similar, but now we need to shift both the homological and q-gradings. Our notation for this is as follows. For $n, m \in \mathbb{Z}$, we define $t^m q^n CKh(D)$ to be the bigraded chain complex whose i, jth group is $CKh^{i-m,j-n}(D)$. The notation is chosen so that

$$P(t^m q^m Kh(D)) = t^m q^n P(Kh(D)).$$

If o is an orientation on our diagram D, we let $n_\pm(D, o)$ be the number of positive/negative crossings in D, and define

$$CKh(D, o) = t^{-n_-(D,o)} q^{n_+(D,o) - 2n_-(D,o)} CKh(D).$$

Exercise 3.6.1. Let D_o be the diagram obtained by taking the oriented resolution at each crossing of D. Show that in the shifted complex $\mathrm{CKh}(D, o)$, the summand $\mathcal{A}(D_o)$ is in homological grading 0.

The first main result of Khovanov's original paper is

Theorem 3.6.2 ([39]). *If* (D, o) *and* (D', o') *are oriented diagrams related by a Reidemeister move, then* $\mathrm{CKh}(D, o)$ *is chain homotopy equivalent to* $\mathrm{CKh}(D', o')$.

The proof of this theorem will be given in the next lecture. It justifies

Definition 3.6.3. If L is an oriented link in S^3 represented by a planar diagram D with orientation o, then $\mathrm{Kh}(L) := \mathrm{Kh}(D, o)$.

The second main result of [39] is that the graded Euler characteristic of $\mathrm{Kh}(L)$ is given by the Jones polynomial:

Theorem 3.6.4. *For any oriented link* L *as above ,*

$$\chi(\mathrm{Kh}(L)) = \overline{V}(L)\,.$$

Proof. Up to overall shifts, this follows easily from equation 1.4.1, which expresses $\langle D\rangle$ as a sum over vertices in the cube of resolutions. We have

$$\begin{aligned}\chi(\mathrm{CKh}(D)) &\sim \sum_v (-q)^{|v|}\,\mathrm{qdim}\,\mathcal{A}(D_v)\\ &\sim \sum_v (-q)^{|v|}(q+q^{-1})^{|D_v|}\\ &\sim \sum_v A^{n-2|v|}B^{|D_v|} = \langle D\rangle\end{aligned}$$

where $q = -A^{-2}$ and $B = -A^2 - A^{-2} = q + q^{-1}$. We leave the reader to check that the overall shifts match up correctly. □

3.7. Functoriality One important property of the Khovanov homology is the fact that it is projectively functorial: cobordisms of links induce maps on Kh which are well-defined up to an overall sign.

Definition 3.7.1. Suppose that L_0 and L_1 are oriented links in $\mathbb{R}^3$. A *cobordism* from L_0 to L_1 is an smooth properly embedded oriented surface $\Sigma \subset \mathbb{R}^3 \times I$ such that $\partial\Sigma = -L_0 \times \{0\} \cup L_1 \times \{1\}$.

Two cobordisms Σ_0, Σ_1 from L_0 to L_1 are equivalent if there is a smooth isotopy $\Phi : \Sigma \times I \to \mathbb{R}^3 \times I$ such that the restriction of Φ to each $\Sigma \times t$ is a cobordism from L_0 to L_1, $\Phi|_{\Sigma\times 0} = \Sigma_0$, $\Phi|_{\Sigma\times 1} = \Sigma_1$, and for each $x \in \partial\Sigma$, $\Phi|_{x\times I}$ is a constant map. (In other words, the isotopy fixes both L_0 and L_1).

As we did with abstract cobordisms, we can form a category whose objects are links in $\mathbb{R}^3$ and whose morphisms are equivalence classes of cobordisms between them. Following [39], we would like to define a functor from this category to the category of abelian groups and linear maps, which on objects is given by Kh.

The basic philosophy behind this definition is to divide the given cobordism into a sequence of elementary cobordisms. We then define the map induced by each elementary cobordism at the chain level, as a map on CKh.

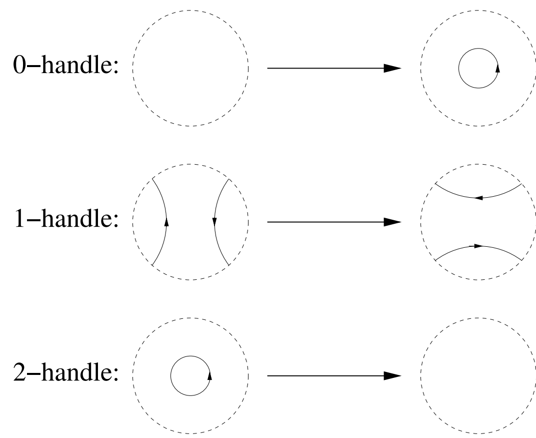


Figure 3.7.2. The three Morse moves.

Definition 3.7.3. A *movie* is a finite subset $S = \{t_1, \dots, t_n\} \subset (0,1)$, together with a planar diagram D_t for each $t \in I - S$, such that for $t \in (t_i, t_{i+1})$, D_t is an isotopy of planar diagrams, and for small ε, the diagrams $D_{t_i-\varepsilon}, D_{t_i+\varepsilon}$ are related by either a Reidemeister move or one of the *Morse moves* shown in Figure 3.7.2.

Let $\Sigma : L_0 \to L_1$ be a cobordism. After a small isotopy of Σ, we may arrange that $L_t := \Sigma \cap (\mathbb{R}^3 \times \{t\})$ is a link for all but finitely many values of t, and that the associated planar diagrams D_t form a movie, which is said to represent Σ. Just as any two planar diagrams representing a given link are related by Reidemeister moves, it can be shown that any two movies representing the same cobordism are related by a finite sequence of elementary moves, known as *movie moves* [12].

To each Reidemeister move R_i $(i = 1, 2, 3)$ relating diagrams D_0 and D_1, we assign a chain map $\Phi_{R_i} : \mathrm{CKh}(D_0) \to \mathrm{CKh}(D_1)$. Φ_{R_i} is the chain homotopy equivalence provided by theorem 3.6.2. These maps will be described explicitly in the next lecture.

Similarly, for each Morse move M_i $(i = 0, 1, 2)$ relating diagrams D_0 and D_1, we assign a chain map $\Phi_{M_i} : \mathrm{CKh}(D_0) \to \mathrm{CKh}(D_1)$. Φ_{M_i} is given by the corresponding map in the TQFT $\mathcal{A}$. To be precise, the crossings of D_0 can naturally be identified with those of D_1, so the vertices of the cube of resolutions for D_0 and D_1 can also be identified. For each vertex ν, the Morse move gives a natural cobordism (corresponding to the addition of a single handle) $S_\nu : (D_0)_\nu \to (D_1)_\nu$. For $x \in \mathcal{A}((D_0)_\nu)$, we define

$$\Phi_{M_i}(x) = \mathcal{A}(S_\nu)(x).$$

We leave it as an exercise to the reader to check that Φ_{M_i} is a chain map.

To a movie $\{D_t\}$, we want to assign a chain map

$$\Phi_{D_t} : \mathrm{CKh}(D_0) \to \mathrm{CKh}(D_1).$$

As t varies in the interval (t_i, t_{i+1}), the diagrams D_t change only by isotopy, so the chain complexes $\mathrm{CKh}(D_t)$ are all isomorphic. The diagrams $D_{t_i-\varepsilon}$ and $D_{t_i+\varepsilon}$ are related by either a Reidemeister or a Morse move, and we then let $\Phi_{t_i} : \mathrm{CKh}(D_{t_i-\varepsilon}) \to \mathrm{CKh}(D_{t_i+\varepsilon})$ be the map defined above. Finally, we let Φ_{D_t} be their composition:

$$\Phi_{D_t} := \Phi_{t_n} \circ \Phi_{t_{n-1}} \circ \cdots \circ \Phi_{t_1}.$$

To sum up, we have assigned a chain map $\Phi_{D_t} : \mathrm{CKh}(D_0) \to \mathrm{CKh}(D_1)$ to each movie D_t, which induces a map on homology $\Phi_{D_t} : \mathrm{Kh}(D_0) \to \mathrm{Kh}(D_1)$. These maps were defined in Khovanov's first paper [39]. The question of whether these maps are functorial (that is, invariant under the movie moves) was first investigated by Jacobsson [35].

Theorem 3.7.4 ([35])**.** *The map Φ_{D_t} is projectively functorial; that is, if D_t and D'_t are two movies representing equivalent cobordisms, then Φ_{D_t} is chain homotopic to $\pm\Phi_{D'_t}$.*

The sign ambiguity is necessary: Jacobsson found explicit pairs of movies representing the same cobordism for which the induced maps have opposite sign. Subsequently, Clark, Morrison and Walker [15] and Blanchet [10] showed that the sign ambiguity can be eliminated, but at the cost of redefining the Khovanov homology.

Theorem 3.7.5. [10, 15] *There is a functor $\mathcal{F}$ from the category of oriented links and cobordisms between them to the category of abelian groups such that*

(1) A planar diagram D for L yields an isomorphism $\iota_D : \mathrm{Kh}(D) \to \mathcal{F}(L)$.
(2) If D_t is a movie representing a cobordism Σ, then $\Phi_{D_t} = \pm\iota_{D_1}^{-1} \circ \mathcal{F}(\Sigma) \circ \iota_{D_0}$.

3.8. Deformations To define $\mathrm{CKh}(D)$, we applied the TQFT $\mathcal{A}$ to the cube of resolutions of D. If we apply a different TQFT $\mathcal{A}'$, the result will still be a chain complex, but its homology need not be invariant under the Reidemeister moves. The structure of a (1+1) dimensional TQFT is more or less determined by the multiplicative ring of its underlying Frobenius algebra. For Khovanov homology, this is $A = \mathbb{Z}[X]/X^2$. We have already seen (Exercise 3.4.7) that in order to get a knot invariant $A' = \mathcal{A}'(S^1)$ must have rank 2, so a natural guess is to consider a Frobenius algebra with multiplicative ring $A' = \mathbb{Z}[X]/(X^2 - aX - b)$ for $a, b \in \mathbb{Z}$. (In other words, $m(\mathbf{x} \otimes \mathbf{x}) = a\mathbf{x} + b$.) We can find a Frobenius algebra of this form; the comultiplication is given by

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{1} - a\mathbf{1} \otimes \mathbf{1} \quad \text{and} \quad \Delta(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x} + b\mathbf{1} \otimes \mathbf{1}.$$

It can be shown that any such $\mathcal{A}'$ determines a knot invariant $\mathrm{Kh}_{\mathcal{A}'}(L)$.

Semi-simple TQFT's: The structure of $\mathrm{Kh}_{\mathcal{A}'}(L)$ was largely determined by Lee and Turner [50, 80]. We begin with the observation that if $X^2 - aX - b$ has distinct roots, $\mathcal{A}'$ can be written in a particularly simple form.

Definition 3.8.1. Suppose $\mathcal{A}'$ is a $1+1$ dimensional TQFT defined over a field F, and let A be the corresponding Frobenius algebra. We say $\mathcal{A}'$ is *semi-simple* if $A' = \bigoplus_{i=1}^{n} A_i$, where each A_i is a 1-dimensional Frobenius algebra.

The existence of a unit and co-unit imply that if $\mathbf{v}_i$ is a nonzero element of A_i, then $m(\mathbf{v}_i \otimes \mathbf{v}_i) = k_i \mathbf{v}_i$ and $\Delta(\mathbf{v}_i) = c_i \mathbf{v}_i \otimes \mathbf{v}_i$, where $c_i \neq 0 \neq k_i$.

Lemma 3.8.2. *Suppose $\mathcal{A}'$ is a $1+1$ dimensional TQFT whose Frobenius algebra has multiplicative ring $A' = F[X]/(p(X))$, where $p(X)$ splits over F with distinct roots. Then $\mathcal{A}'$ is semi-simple.*

Proof. Let $n = \dim A' = \deg p$. We view multiplication by X as a linear operator $X : A' \to A'$. This operator satisfies the relation $p(X) = 0$, so p divides the characteristic polynomial of X. But $n = \deg p$, so p must be the characteristic polynomial. Since p splits over F, we have distinct eigenvalues $\lambda_1, \ldots, \lambda_n \in F$. It follows that X is diagonalizable. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of eigenvectors with $X(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$.

Let A_i be the subspace of A' spanned by $\mathbf{v}_i$. We will show that $A' \simeq \oplus A_i$ (as Frobenius algebras). First consider the multiplication. We have

$$X(m(\mathbf{v}_i \otimes \mathbf{v}_j)) = m(m(X \otimes \mathbf{v}_i) \otimes \mathbf{v}_j) = \lambda_i m(\mathbf{v}_i \otimes \mathbf{v}_j),$$

and

$$X(m(\mathbf{v}_i \otimes \mathbf{v}_j)) = X(m(\mathbf{v}_j \otimes \mathbf{v}_i)) = \lambda_j m(\mathbf{v}_j \otimes \mathbf{v}_i) = \lambda_j m(\mathbf{v}_i \otimes \mathbf{v}_j).$$

Since $\lambda i \neq \lambda_j$, we must have $m(\mathbf{v}_i \otimes \mathbf{v}_j) = 0$ for $i \neq j$. Similarly, we must have $X(m(\mathbf{v}_i \otimes \mathbf{v}_i)) = \lambda_i m(\mathbf{v}_i \otimes \mathbf{v}_i)$, which implies $m(\mathbf{v}_i \otimes \mathbf{v}_i) \in A_i$.

Next, the comultiplication: on $A' \otimes A'$ we have two linear operators X_l and X_r which act by multiplication by X on the left and right-hand tensor factors. $\Delta(\mathbf{v}_i)$ is an eigenvector for both X_l and X_r; the eigenvalue in both cases is λ_i. It follows that $\Delta(\mathbf{v}_i) = c_i \mathbf{v}_i \otimes \mathbf{v}_i$. □

In contrast, the TQFT $\mathcal{A}$ used to define Khovanov homology is not semi-simple; the Jordan normal form of the operator X is given by the upper triangular matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Canonical Generators: Let D be a diagram representing L. If $\mathcal{A}'$ is semi-simple, Lee describes an explicit set of generators for $Kh_{\mathcal{A}'}(D)$. The elements of this set are in bijection with orientations on D.

Given an orientation o of D, the oriented resolution of D with respect to o is an oriented collection of circles in the plane. Let $h(C)$ be the nesting height (as in the proof of Lemma 2.6) of such a circle, and let

$$p(C) = \begin{cases} h(C) & \text{if } C \text{ is oriented counterclockwise} \\ h(C)+1 & \text{if } C \text{ is oriented clockwise} \end{cases}$$

Definition 3.8.3. Suppose $\mathcal{A}'$ is semi-simple, with simple summands generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. If o is an orientation of D, the *canonical generator* $\mathfrak{s}_o \in CKh_{\mathcal{A}'}(D)$ is state obtained by taking the oriented resolution of D with respect to o and labeling each circle C by $\mathbf{v}_1$ if $p(C)$ is odd, and $\mathbf{v}_2$ if $p(C)$ is even.

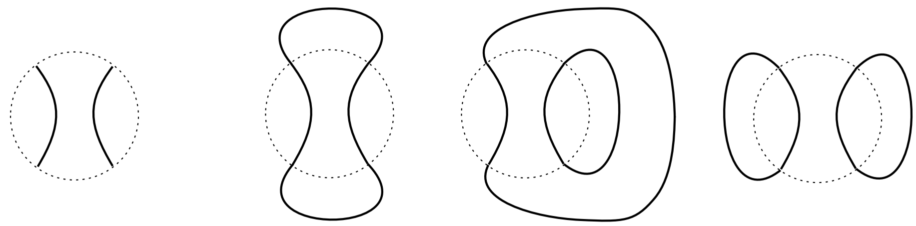


Figure 3.8.4. On the left, two strands in a disk. On the right are some possible closures.

Lemma 3.8.5. *Consider an embedded disk which intersects the diagram for $\mathfrak{s}_o$ in two strands, as shown on the left of Figure 3.8.4. If the orientations on the two strands are parallel, they have different labels; if the orientations are opposite, the labels are the same.*

Proof. Consider the circles containing the two strands. Up to a rotation of the diagram, they must be in one of the three configurations shown on the right-hand side of the figure. If there is one circle in the closure, the two orientations must point in opposite directions and have the same label. If there are two nested circles C_1 and C_2, then $h(C_1) \equiv h(C_2) + 1 \mod 2$. In this situation, the orientations are either parallel, and then $p(C_1) \not\equiv p(C_2)$, or opposite, and then $p(C_1) \equiv p(C_2)$. Either way, the statement holds. The case where C_1 and C_2 are not nested is similar. □

Corollary 3.8.6. $\mathfrak{s}_o$ *is closed in* $CKh_{\mathcal{A}'}(D)$.

Proof. Let v_o be the vertex of the cube of resolutions associated with the oriented resolution of L, and let $e : v_o \to v'$ be an edge. The two strands near the crossing have parallel orientations. By the lemma, they have opposite labels. Thus they belong to different components of D_{v_o}, and the cobordism S_e is a merge. Since $m(\mathbf{v}_1 \otimes \mathbf{v}_2) = 0$, we have $\mathcal{A}'(S_e)(\mathfrak{s}_o) = 0$. □

Theorem 3.8.7. *(*[50, 80]*) If $\mathcal{A}'$ is a semisimple TQFT defined over F, then $Kh_{\mathcal{A}'}(L; F)$ is freely generated by the classes $\mathfrak{s}_o$, where o runs over the set of orientations on L. In particular, $Kh_{\mathcal{A}'}(L; F)$ has dimension $2^{|L|}$.*

Proof. By induction on the number of crossings in D. When D has no crossings, $Kh_{\mathcal{A}'}(D) = \mathcal{A}'(D)$ has dimension $2^{|D|}$ and is generated by states in which we label each component of D by either $\mathbf{v}_1$ or $\mathbf{v}_2$. These are the canonical generators.

In general, chose a crossing c of D and let D_0 and D_1 be the diagrams obtained by resolving c. There is a short exact sequence

$$0 \to CKh_{\mathcal{A}'}(D_1) \to CKh_{\mathcal{A}'}(D) \to CKh_{\mathcal{A}'}(D_0) \to 0$$

which comes from splitting the cube of resolutions for D into two halves, according to the way that c is resolved. We will compute the boundary map $\partial : \mathrm{Kh}_{\mathcal{A}'}(D_0) \to \mathrm{Kh}_{\mathcal{A}'}(D_1)$ in the associated long exact sequence.

By induction, $\mathrm{Kh}_{\mathcal{A}'}(D_0)$ and $\mathrm{Kh}_{\mathcal{A}'}(D_1)$ are generated by canonical generators. Let o be an orientation on D_0, and let ν_o be the vertex of the cube of resolutions for D_0 given by taking the oriented resolution with respect to o. The canonical generator $\mathfrak{s}_o \in \mathrm{CKh}(D_0)$ is supported at ν_o. In the cube of resolutions for D, there is a unique edge e which points from ν_o to a vertex ν_o' in the cube of resolutions for D_1; namely, the edge corresponding to changing the resolution at c from the 0 resolution to the 1 resolution. It follows that $\partial[\mathfrak{s}_o] = [\mathcal{A}'(S_e)(\mathfrak{s}_o)]$.

At this point, the argument splits into three cases, depending on how the ends at the crossing c are joined up in L. The three possibilities are shown in Figure 3.8.8 below.

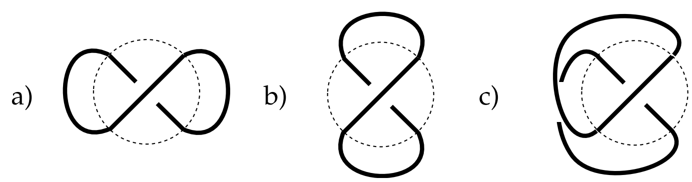
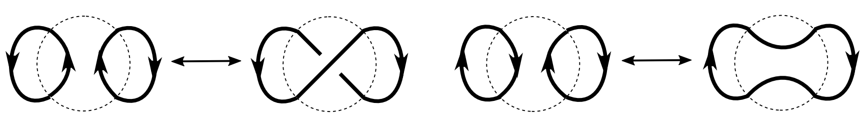


Figure 3.8.8. The three possible ways of joining up the ends of a crossing.

In case a), $|D| = |D_1| = |D_0| - 1$, so there are twice as many orientations on D_0 as on D or D_1. We can divide the orientations on D_0 into two types: those in which the orientation on the two strands near c is parallel, and those in which it is opposite. As shown in the figure below, orientations of the first type are in bijection with orientations on D, and orientations of the second type are in bijection with orientations on D_1.



If o is of the first type, Lemma 3.8.5 implies that the labels on the two strands are different. It follows that S_e is a merge, and $\mathcal{A}'(S_e)(\mathfrak{s}_o) = 0$. If o is of the second type, the labels are the same. In this case, $\mathcal{A}'(S_e)(\mathfrak{s}_o) = c\mathfrak{s}_{o_1}$, regardless of whether S_e is a merge or a split. Here o_1 is the orientation on D_1 which corresponds to o, and $c \neq 0$. We conclude that $\partial : \mathrm{Kh}_{\mathcal{A}'}(D_0) \to \mathrm{Kh}_{\mathcal{A}'}(D_1)$ is surjective, and $\mathrm{Kh}_{\mathcal{A}'}(D) = \ker \partial$ is freely generated by the classes $[\mathfrak{s}_o]$, where o runs over orientations of D.

Cases b) and c) are very similar. In case b), D_1 has twice as many orientations as D or D_0, and the boundary map is a surjection, while in case c), D has twice as many orientations as D_0 and D_1, and the boundary map vanishes. We leave the details for the reader. □

In order to fix the homological grading on $\mathrm{Kh}(L)$, we needed to fix an orientation o on L. The vertex D_o of the cube of resolutions is in homological grading 0. If K is a knot either orientation gives the same homological grading, and both canonical generators are in homological grading 0. More generally, the two generators $\mathfrak{s}_o, \mathfrak{s}_{-o}$ will be in homological grading 0; the grading of all the other generators is determined by the pairwise linking numbers of the components of L [50].

Lee's spectral sequence: The existence of the q-grading on Khovanov homology depended on the fact that $\mathcal{A}$ is a graded TQFT: $\widetilde{q}(\mathcal{A}(S)(x)) = \widetilde{q}(x) + \chi(S)$. A quick look at the formulas for m' and Δ' shows that this does not hold for $\mathcal{A}'$; however $\mathcal{A}'$ is *filtered*, in the sense that the $\widetilde{q}$-grading of every term that appears in $\mathcal{A}(S)(x)$ is at least $\widetilde{q}(x) + \chi(S)$. The associated graded TQFT (obtained by ignoring all terms that strictly raise the q-grading) is $\mathcal{A}$. In turn, $\mathrm{CKh}_{\mathcal{A}'}(D)$ is a filtered chain complex. The associated graded chain complex (obtained by ignoring all terms in the differential that strictly raise the q-grading) is $\mathrm{CKh}(D)$. We deduce from these observations that

Proposition 3.8.9. *There is a spectral sequence with* E_1 *term* $\mathrm{Kh}(L)$ *which abuts to* $\mathrm{Kh}_{\mathcal{A}'}(L)$*. The filtration grading in this spectral sequence is given by the* q*-grading on* $\mathrm{Kh}(L)$*.*

We should think of the combination of Theorem 3.8.7 and Proposition 3.8.9 and as categorifying the fact that $V_L(1) = 2^{|L|}$. This is the first incidence of a general principle: identities where we specialize a variable in a polynomial typically categorify to give a spectral sequence. In this case (and in many others) it can be shown that the E_i term of the sequence is an invariant of L for all $i > 0$. When this is the case we get a bonus invariant: the filtration grading of the surviving generators in the spectral sequence.

For concreteness, we now specialize to the situation originally considered by Lee, in which $F = \mathbb{Q}$ and $A' = \mathbb{Q}[X]/(X^2 - 1)$. In this case, it can be shown [66] that the filtration grading of the two generators in the spectral sequence for $\mathrm{CKh}_{\mathcal{A}'}(K;\mathbb{Q})$ are of the form s_{min}, s_{max}, where $s_{max} - s_{min} = 2$

Definition 3.8.10. If K is a knot in S^3, $s(K) = s_{min} + 1 = s_{max} - 1$.

For example if $\mathrm{Kh}(\bigcirc) = \mathcal{A}(S^1)$ has dimension 2, so the spectral sequence has already converged at the E_1 term. In this case $s_{min} = -1, s_{max} = 1$, and $s(\bigcirc) = 0$.

Maps induced by cobordism: If $\Sigma : L_0 \to L_1$ is a cobordism, we can define a map $\Phi'_\Sigma : \mathrm{Kh}_{\mathcal{A}'}(L_0) \to \mathrm{Kh}_{\mathcal{A}'}(L_1)$ exactly as we did for Khovanov homology, but using $\mathcal{A}'$ in place of $\mathcal{A}$. Φ'_Σ is a filtered map of degree $\chi(\Sigma)$.

If $\widehat{o}$ is an orientation on Σ, let $-\widehat{o}_0$ and $\widehat{o}_1$ be the induced orientations on L_0 and L_1. (With this convention, if $\Sigma = L \times [0,1]$, $\widehat{o}_0$ and $\widehat{o}_1$ are the same orientation on L.) Let $\mathcal{O}(\Sigma)$ be the set of all orientations on Σ.

Theorem 3.8.11 ([66]). *If* o *is an orientation on* L_0 *and* $\Sigma : L_0 \to L_1$ *is a cobordism, then*

$$\Phi'_\Sigma(\mathfrak{s}_o) = \sum_{\{\widehat{o}\in\mathcal{O}(\Sigma)\,|\,\widehat{o}_0=o\}} c_{\widehat{o}}\mathfrak{s}_{\widehat{o}_1}$$

where $c_{\widehat{o}} \neq 0$ *for all orientations* $\widehat{o} \in \mathcal{O}(\Sigma)$*.*

Sketch of Proof. First, note that

$$\mathcal{O}(\Sigma \circ \Sigma') = \{(\widehat{o}, \widehat{o}') \in \mathcal{O}(\Sigma) \times \mathcal{O}(\Sigma') \,|\, \widehat{o}_0 = \widehat{o}'_1\}.$$

If the statement holds for Σ and Σ', then

$$\begin{aligned}\Phi'_{\Sigma\circ\Sigma'}(\mathfrak{s}_o) &= \Phi'_\Sigma \circ \Phi'_{\Sigma'}(\mathfrak{s}_o)\\ &= \sum_{\{\widehat{o}\in\mathcal{O}(\Sigma)\,|\,\widehat{o}_0=\widehat{o}'_1\}} \sum_{\{\widehat{o}'\in\mathcal{O}(\Sigma')\,|\,\widehat{o}'_0=o\}} c_{\widehat{o}}c_{\widehat{o}'}\mathfrak{s}_{\widehat{o}_1}\\ &= \sum_{\{\widehat{o}''\in\mathcal{O}(\Sigma\circ\Sigma')\,|\,\widehat{o}''_0=o\}} c_{\widehat{o}''}\mathfrak{s}_{\widehat{o}''_1},\end{aligned}$$

so it holds for $\Sigma \circ \Sigma'$ as well. Hence it is enough to check that the statement holds for elementary cobordisms. The case where Σ is cobordism corresponding to a Reidemeister move is checked in [66].

We check the statement holds for Morse moves. Let o be an orientation on L_0. First, suppose Σ is a 0-handle attachment, and let C be the new circle created in L_0. There are two orientations on Σ compatible with o, corresponding to the two possible orientations on C. Now $\Phi'_\Sigma(\mathfrak{s}_o)$ is the state in which all the labels on the old circles remain the same and C is labeled by 1. Since $1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ with both $c_i \neq 0$, the statement holds.

Next, suppose Σ is obtained by attaching a 1-handle. The 1-handle attachment can either be compatible with o or incompatible. If it is compatible, there is a unique orientation $\widehat{o} \in \mathcal{O}(\Sigma)$ with $\widehat{o}_0 = o_0$. The two strands on which the attachment takes place have opposite orientations in $\mathfrak{s}_o$, hence the same label, so we have $\Phi'(\Sigma)(\mathfrak{s}_o) = c\mathfrak{s}_{\widehat{o}_1}$ with $c \neq 0$ regardless of whether the cobordism is a split or a merge.

If the orientation on the 1-handle is incompatible with o, then there is no $\widehat{o} \in \mathcal{O}(\Sigma)$ with $\widehat{o}_0 = o$. The two strands on which the attachment takes place have the same orientation in $\mathfrak{s}_0$ hence opposite labels. It follows that the cobordism is a merge and $\Phi'(\Sigma)(\mathfrak{s}_o) = 0$. Thus the statement holds in this case.

Finally, we consider the case where Σ is a 2-handle attachment. In this case, there is a unique $\widehat{o} \in \mathcal{O}(\Sigma)$ with $\widehat{o}_0 = o$. Since $\varepsilon'(\mathbf{v}_1) \neq 0 \neq \varepsilon'(\mathbf{v}_2)$, it is easy to see that $\Phi'_\Sigma(\mathfrak{s}_o) = c\mathfrak{s}_{\widehat{o}_1}$ with $c \neq 0$, and the statement holds in this case as well. □

Corollary 3.8.12. *If* K_0 *and* K_1 *are knots and* $\Sigma : K_0 \to K_1$ *is a connected cobordism, then* $\Phi'_\Sigma : Kh_{\mathcal{A}'}(K_0; \mathbb{Q}) \to Kh_{\mathcal{A}'}(K_1; \mathbb{Q})$ *is an isomorphism.*

Corollary 3.8.13. *If* $\Sigma : K_0 \to K_1$ *is a connected cobordism, then* $s(K_1) \geqslant s(K_0) + \chi(\Sigma)$*.*

Proof. If D is a diagram of K we have

$$s_{\max}(K) = \max\{q(x) \mid [x] \neq 0 \in Kh'(D)\}.$$

Let D_0 be a diagram of K_0, and choose an $x \in CKh'(D_0)$ which realizes the maximum. Then $[\Phi_\Sigma(x)] \neq 0 \in Kh'(K_1)$, so

$$s_{\max}(K_1) \geqslant q(\Phi_\Sigma(x)) \geqslant q(x) + \chi(\Sigma) = s_{\max}(K_0) + \chi(\Sigma). \qquad \square$$

The slice genus:

Definition 3.8.14. If K is a knot in S^3, the *smooth slice genus* of K is

$$g_*(K) = \min\{g(\Sigma) \mid \Sigma \hookrightarrow B^4 \text{ is a smoothly embedded, orientable, and } \partial\Sigma = K\}$$

If Σ is a surface as in the definition, removing a small ball around a point in Σ gives a cobordism $\widetilde{\Sigma} : K \to \bigcirc$ with $\chi(\widetilde{\Sigma}) = -2g(\Sigma)$. Applying Corollary 3.8.13, we get $s(K) \leqslant 2g(\Sigma) + g(\bigcirc) = 2g(\Sigma)$.

Exercise 3.8.15. Using exercise 3.4.5, show that $s(\overline{K}) = -s(\overline{K})$.

Since $g_*(K) = g_*(\overline{K})$, we deduce

Corollary 3.8.16. $|s(K)| \leqslant 2g_*(K)$.

Unlike the Seifert genus, the slice genus can vanish for a nontrivial knot. If $g_*(K) = 0$, we say that K is *slice*.

Exercise 3.8.17. Show that $g_*(K\#\overline{K}) = 0$ for any knot K.

A surface $\Sigma \subset B^4$ is *locally flat* if every point of $\Sigma \subset B^4$ has an open neighborhood U such that the pair $(U, \Sigma \cap U)$ is homeomorphic to $(\mathbb{R}^4, \mathbb{R}^2)$. The *topological slice genus* is defined to be

$$g_*^{top}(K) = \min\{g(\Sigma) \mid \Sigma \subset B^4 \text{ is locally flat, orientable, and } \partial\Sigma = K\}.$$

A smooth surface is locally flat, but it turns out that the converse is false.

Theorem 3.8.18 (Freedman). *If* $\Delta(K) = 1$, *then* $g_*^{top}(K) = 0$.

Let $P(-3,5,7)$ be the $(-3,5,7)$ pretzel knot from Figure 1.1.3. As we saw in execise 7, section 2, $\Delta(P(-3,5,7)) = 1$, so this knot is topologically slice. In contrast, direct computation shows that $s(P(-3,5,7)) = 2$ and hence that $g_*(P(-3,5,7)) = g(P(-3,5,7)) = 1$. Hence s can be used to detect the difference between smooth and locally flat embeddings. In combination with a theorem of Freedman and Quinn on smoothing topological 4-manifolds with boundary, this can be used to construct a manifold homeomorphic, but not diffeomorphic, to the standard $\mathbb{R}^4$. For details, see [25].

3.9. Connections and Further Reading There is an extensive literature on Khovanov homology — far more than we can survey here. We mention only a few highlights. First, its module structure. As observed by Khovanov [42], Kh(L) is a module over the ring $\mathbb{Z}[X_1, \ldots, X_n]/(X_1^2, \ldots X_n^2)$, where there is one variable for each component of X. When $L = K$ is a knot, this gives reduced homology groups $\mathrm{Kh}_r(K)$ satisfying $\chi(\mathrm{Kh}_r(K)) = V(K)$ (the normalized Jones polynomial).

Second, its behavior for the simplest knots. If K is an alternating knot, $\mathrm{Kh}(K)$ is *thin*: $\mathrm{Kh}_r^{i,j}(K) = 0$ unless $2i - j = \sigma(K)$. It follows that $\mathrm{Kh}_r(K)$ (and also $\mathrm{Kh}(K)$) is determined by the Jones polynomial and signature of K. This was conjectured (in the unreduced case) by Bar-Natan [3], and proved by Lee [50], who created her epynomous spectral sequence in the process.

Third, its relations with Floer homology. The most immediate comparison is with knot Floer homology $\widehat{\mathrm{HFK}}(K)$, which categorifies the Alexander polynomial and is discussed in Hom's lectures. The invariant $s(K)$ was defined in analogy with the Ozsváth-Szabó τ-invariant and gives a similar bound on the slice genus.

In another direction, there are many invariants which satisfy unoriented skein exact sequences similar to that satisfied by Khovanov homology. If we use $\mathbb{Z}/2$ coefficients, such theories are often the target of spectral sequences whose E_1 term is $\mathrm{Kh}(K;\mathbb{Z}/2)$. The first example is in a beautiful paper of Ozsváth and Szabó [61], who constructed a spectral sequence from $\mathrm{Kh}(K;\mathbb{Z}/2)$ to $\widehat{\mathrm{HF}}(\Sigma(-K);\mathbb{Z}/2)$ — the Heegaard Floer homology of the branched double cover of K. Another is due to Kronheimer and Mrowka [48], who constructed a spectral sequence from $\mathrm{Kh}(K;\mathbb{Z}/2)$ to their instanton knot homology $\mathrm{KHI}(K;\mathbb{Z}/2)$. As a consequence, they proved that Khovanov homology detects the unknot.

Finally, the symplectic Khovanov homology of Seidel and Smith [74] is defined geometrically using Lagrangian Floer homology. It is isomorphic to Khovanov homology, as proved by Abouzaid and Smith [2].

Two more recent developments worth knowing are the Batson-Seed spectral sequence [6], which relates the Khovanov homology of a link to that of its components, and Grigsby-Licata-Wehrli's $\mathfrak{sl}_2$ action on sutured annular Khovanov homology [30].

4. Khovanov Homology for Tangles

In this lecture, we will extend the definition of Khovanov homology to the 2-category of tangles and cobordisms between them. There are many ways to do this, all of them more or less equivalent [5,41,45,77]. We will follow Bar-Natan [5], whose construction is appealingly geometric. The ideas in this lecture will play a central role in the subsequent lectures, so it's worth spending time thinking about them.

4.1. Tangles For each positive integer n, we fix a set X_n of n points in $\mathbb{R}$.

Definition 4.1.1. An (n, m)-tangle is a proper smooth embedding

$$T : \left(\coprod_{i=1}^{a} I\right) \amalg \left(\coprod_{j=1}^{b} S^1\right) \hookrightarrow \mathbb{R}^2 \times I$$

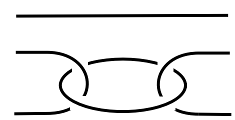
such that the set $\amalg_{i=1}^{a} \partial I$ is mapped bijectively onto $(X_n \times 0 \times 0) \amalg (X_m \times 0 \times 1)$. In particular, $a = (n+m)/2$.

We label the coordinates on $\mathbb{R}^2$ by (x, y), and the coordinate on $[0,1]$ by s. Two tangles T_0, T_1 are equivalent if there is an isotopy

$$\Phi : \left((\amalg_{i=1}^{a} I) \amalg (\amalg_{j=1}^{b} S^1) \right) \times I \to \mathbb{R}^2 \times I$$

such that $T_t := \Phi(\cdot, t)$ is a tangle for all $t \in [0,1]$. Note that this definition implies that the boundary of the tangle is fixed under the isotopy.

Definition 4.1.2. We let $\mathcal{T}_{n,m}$ be the set of (n,m) tangles, and let $\mathbf{T}_{n,m}$ be the set of (n,m) tangles up to isotopy.



For example, $\mathbf{T}_{0,0}$ is the set of isotopy classes of links in $\mathbb{R}^2 \times I$, which can be naturally identified with the set of isotopy classes of links in $\mathbb{R}^3$. Just as with knots and links, a tangle may be represented by a planar diagrams which records the projection of the tangle to the sx plane, together with over and under-crossings. As an example, a planar diagram of a (3,3) tangle is shown in the figure to the right. As with links, it can be shown that two diagrams which represent the same tangle are related by a sequence of Reidemeister moves.

An important property of tangles is that they satisfy a partial composition rule: if $T \in \mathcal{T}_{n,m}$ and $T' \in \mathcal{T}_{m,l}$, then $TT' \in \mathcal{T}_{n,l}$ is the tangle obtained by stacking T and T' horizontally (in the s direction), as shown in the figure below.

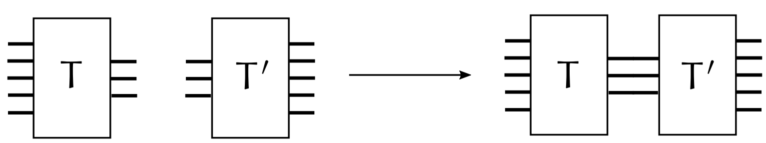


Figure 4.1.3. Horizontal composition of tangles.

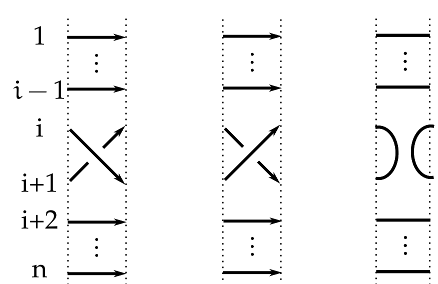


Figure 4.1.4. From left: the elementary braids σ_i σ_i^{-1} and the non-invertible tangle U_i.

Example 4.1.5. It is easy to see that $\mathbf{T}_{n,n}$ forms a monoid under composition. The *braid group on* n *strands* (written Br_n) is the submonoid of $\mathbf{T}_{n,n}$ generated by the *elementary braids* $\sigma_i^{\pm 1}$ shown in Figure 4.1.4. The reader should convince him/herself that Br_n is indeed a group. (Note that $\mathbf{T}_{n,n}$ is not a group; for example the tangle U_i shown in the figure is not invertible.)

It can be shown that Br_n has a presentation

$$\mathrm{Br}_n = \left\langle \sigma_1, \dots \sigma_{n-1} \middle| \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle .$$

The first set of relations is known as *far-commutativity* while the second corresponds to the third Reidemeister move.

Definition 4.1.6. If T_0, T_1 are (n, m) tangles, a *cobordism* $\Sigma : T_0 \to T_1$ is a smooth properly embedded orientable surface $\Sigma \subset \mathbb{R}^2 \times I \times I$ such that

$$\partial\Sigma = T_0 \times 0 \cup T_1 \times 1 \cup (X_n \times 0 \times 0 \times I) \cup (X_m \times 0 \times 1 \times I).$$

We say cobordisms $\Sigma, \Sigma' : T_0 \to T_1$ are equivalent if there is a diffeomorphism $\varphi : \mathbb{R}^2 \times I \times I \to \mathbb{R}^2 \times I \times I$ such that φ restricts to the identity on $\partial(\mathbb{R}^2 \times I \times I)$ and $\varphi(\Sigma) = \Sigma'$.

We will denote the coordinates on $\mathbb{R}^2 \times I \times I$ by (x, y), s and t. Tangle cobordisms can be composed in two different ways. First suppose $\Sigma : T_0 \to T_1$ is a cobordism of (n, m) tangles, and $\Sigma : T_0' \to T_1'$ is a cobordism of (m, l) tangles. Then $\Sigma\Sigma' : T_0T_0' \to T_1T_1'$ is the cobordism obtained by stacking Σ and Σ' in the s direction. This is called *horizontal composition*. Second, if T_0, T_1, T_2 are all (m, n) tangles, $\Sigma_0 : T_0 \to T_1$, and $\Sigma_1 : T_1 \to T_2$, we get a cobordism $\Sigma_0 \circ \Sigma_1 : T_0 \to T_2$ by stacking in the t direction. This is called *vertical composition*. It is not hard to see that both of these notions are well-behaved with respect to equivalence of cobordisms. (Tangle cobordisms are hard to draw, but see Figures 4.2.4 and 4.2.5, which illustrate the analogous compositions for planar tangles.)

A good way to think about these various compositions is to say that there is a category whose objects are the sets X_n for $n \geqslant 0$ and for which $\mathrm{Mor}(X_n, X_m)$ is the set of (n, m) tangles. For each n and m, the set $\mathrm{Mor}(X_n, X_m)$ is itself the set of objects of a category whose morphisms are equivalence classes of cobordisms. A more sophisticated way of saying this is that we have a 2-*category* with objects X_n ($n \geqslant 0$), 1- morphisms given by tangles, and 2-morphisms given by equivalence classes of cobordisms between tangles.

Exercise 4.1.7. Show that if T_0 and T_1 are isotopic (n, m) tangles, they are isomorphic as objects of the category $\mathrm{Mor}(X_n, X_m)$.

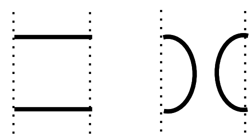
4.2. Planar Tangles

Definition 4.2.1. A *planar* (n, m) *tangle* is a proper smooth embedding

$$T : \left(\coprod_{i=1}^{a} I \right) \amalg \left(\coprod_{j=1}^{b} S^1 \right) \hookrightarrow \mathbb{R} \times I$$

such that the set $\amalg_{i=1}^{a} \partial I$ is mapped bijectively onto $(X_n \times 0) \amalg (X_m \times 1)$.

We denote the set of all such tangles by $\mathcal{P}_{n,m}$. We say a planar tangle is *simple* if it has no closed components, and let $\mathbf{P}_{n,m}$ be the set of isotopy classes of simple planar (n, m) tangles. For example, $\mathbf{P}_{2,2}$ consists of the two tangles $\mathbf{1}$ and U shown in the figure below.



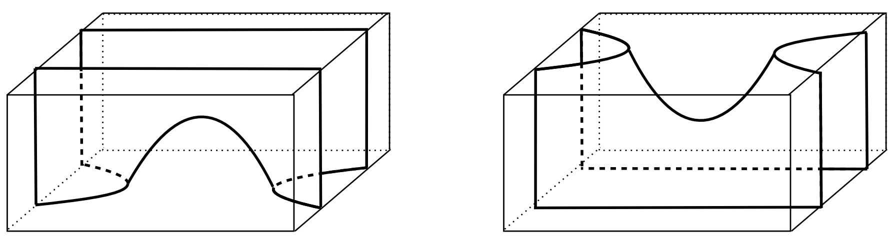
$\mathbf{1}$ $\qquad$ U

Exercise 4.2.2. Show that $\mathbf{P}_{n,m}$ has $C_{(n+m)/2}$ elements, where $C_r = \dfrac{1}{r+1}\dbinom{2r}{r}$ is the r*th Catalan number*.

Definition 4.2.3. If T_0, T_1 are planar (n, m) tangles, a *cobordism* $\Sigma : T_0 \to T_1$ is a smooth properly embedded orientable surface $\Sigma \subset \mathbb{R} \times I \times I$ such that

$$\partial\Sigma = T_0 \times 0 \cup T_1 \times 1 \cup (X_n \times 0 \times I) \cup (X_m \times 1 \times I).$$

We say cobordisms $\Sigma, \Sigma' : T_0 \to T_1$ are equivalent if there is a diffeomorphism $\varphi : \mathbb{R} \times I \times I \to \mathbb{R} \times I \times I$ such that φ restricts to the identity on $\partial(\mathbb{R} \times I \times I)$ and $\varphi(\Sigma) = \Sigma'$.



$S : U \to \mathbf{1}$ $\qquad$ $S' : \mathbf{1} \to U$

For example, the figure above shows the saddle cobordism $S : U \to \mathbf{1}$ and its reverse $S' : \mathbf{1} \to U$.

As before, if we are given $\Sigma : T_0 \to T_1$ a cobordism of planar (n, m) tangles and $\Sigma : T_0' \to T_1'$ a cobordism of planar (m, l) tangles, then we can form their horizontal composition $\Sigma\Sigma' : T_0 T_0' \to T_1 T_1'$ by stacking in the s direction, as illustrated in the figure below:

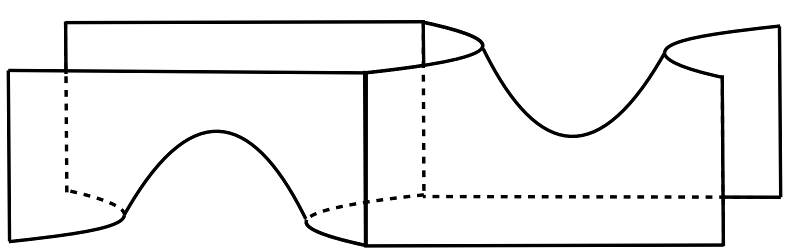


Figure 4.2.4. The horizontal composition $SS' : U\mathbf{1} \to \mathbf{1}U$.

If T_0, T_1, T_2 are all planar (m,n) tangles, $\Sigma_0 : T_0 \to T_1$, and $\Sigma_1 : T_1 \to T_2$, their vertical composition $\Sigma_0 \circ \Sigma_1 : T_0 \to T_2$ is obtained by stacking in the t direction:

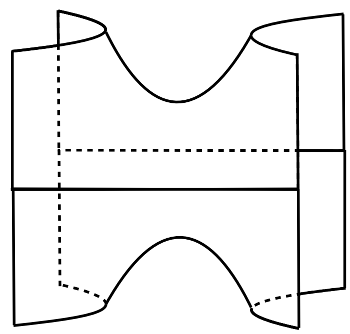


Figure 4.2.5. The vertical composition $S \circ S' : U \to U$.

4.3. The Kauffman Bracket Let $R = \mathbb{Z}[q^{\pm 1}]$, and define $V_{n,m}$ to be the free R-module generated by $\mathbf{P}_{n,m}$. Suppose D is a planar diagram representing an (n,m) tangle. If D has k crossings, there will be 2^k ways to resolve all k, and the set of possible resolutions is in bijection with the vertices of $[0,1]^k$. The diagram D_ν corresponding to the resolution at vertex ν is a planar (n,m) tangle.

Definition 4.3.1. For D an (n,m) tangle diagram, its Kauffman bracket is

$$\langle D \rangle = \sum_{\nu} (-q)^{|\nu|}(q+q^{-1})^{|D_\nu|}\widetilde{D_\nu} \in V_{n,m}$$

where $|D_\nu|$ denotes the number of closed components of D_ν, and $\widetilde{D_\nu}$ is the simple planar tangle obtained by erasing all closed components of D_ν.

Exercise 4.3.2. Let R_i and R_i' be the (i,i) tangle diagrams corresponding to the ith Reidemeister move. Check that $\langle R_i \rangle \sim \langle R_i' \rangle$, where $p_1 \sim p_2$ means $p_1 = \pm q^k p_2$ for some $k \in \mathbb{Z}$.

Composition: We define $\cdot : V_{n,m} \times V_{m,l} \to V_{n,l}$ by setting $D \cdot D' = \langle DD' \rangle$ when D and D' are simple planar tangles, and extending bilinearly.

Proposition 4.3.3. *For D an (n,m) tangle diagram and D' an (m,l) tangle diagram,*

$$\langle DD' \rangle = \langle D \rangle \cdot \langle D' \rangle.$$

Proof. The cube of resolutions for DD' is the product of the cube of resolutions for D and the cube for D', so

$$\begin{aligned}\langle DD' \rangle &= \sum_{(\nu,\nu')} (-1)^{|\nu|+|\nu'|}(q+q^{-1})^{|D_\nu D'_{\nu'}|}[\widetilde{D_\nu D_{\nu'}}] \\ &= \sum_{(\nu,\nu')} (-1)^{|\nu|+|\nu'|}(q+q^{-1})^{|D_\nu|+|D'_{\nu'}|}\widetilde{D_\nu} \cdot \widetilde{D_{\nu'}} \\ &= \langle D \rangle \langle D' \rangle.\end{aligned}$$

□

Proposition 4.3.4. *If two planar diagrams* D *and* D' *represent the same tangle, then we have* $\langle D\rangle \sim \langle D'\rangle$.

Proof. We checked in the exercise above that the statement holds for the planar diagrams R_i and R_i' corresponding to the ith Reidemeister move. From this, it is easy to see that the statement holds if D_r and D_r' are (n,n) tangle diagrams obtained by adding some horizontal arcs to a pair of Reidemeister diagrams. Finally, we observe that if D and D' are any two tangles related by a Reidemeister move, then we can factor $D = D_0 D_r D_1$ and $D' = D_0 D_r' D_1$ where D_r and D_r' are as above. Then

$$\langle D\rangle = \langle D_0\rangle \cdot \langle D_r\rangle \cdot \langle D_1\rangle = \langle D_0\rangle \cdot (\pm q^k \langle D_r'\rangle) \cdot \langle D_1\rangle = \pm q^k \langle D'\rangle. \qquad \square$$

Note that the proof of invariance of the Jones polynomial that we gave in section 1.4 silently used the multiplicativity property of the bracket that we have made explicit here. In order to eliminate the factor of $\pm q^k$, we would need to consider framed tangles, but we will not do this here.

Definition 4.3.5. The multiplication above makes the R-module $V_{n,n}$ into an algebra, which is the *Temperly-Lieb algebra* TL_n.

TL_n is generated by the elementary planar tangles $U_1, \dots, U_n$ shown in Figure 4.1.4. It has a presentation

$$TL_n = \left\langle U_1, \dots U_{n-1} \;\middle|\; \begin{array}{c} U_i^2 = (q+q^{-1})U_i \\ U_i U_j = U_j U_i \text{ for } |i-j|>1 \\ U_i U_j U_i = U_i \text{ for } |i-j| = 1 \end{array} \right\rangle.$$

Exercise 4.3.6. Check that the relations above hold. Write the five simple planar $(3,3)$ tangles as products of the generators U_1, U_2, and compute the effects of multiplying each of these basis elements by U_1 and U_2.

Exercise 4.3.7. Show that there is a homomorphism $\psi_2 : Br_n \to TL_n$ given by

$$\psi_2(\sigma_i) = 1 - qU_i \qquad \psi_2(\sigma_i^{-1}) = 1 - q^{-1}U_i,$$

and that $\langle\sigma\rangle \sim \psi_2(\sigma)$. Compute $\psi_2(\sigma_1\sigma_2\sigma_3)$ and $\psi_2(\sigma_1^n)$.

4.4. Category theory The main goal of this lecture is to categorify the Kauffman bracket introduced in the previous section. First, we review some basic notions from category theory.

Gradings: A category $\mathcal{C}$ is *graded* if there is an invertible functor $q : \mathcal{C} \to \mathcal{C}$, called the *grading shift*. The graded categories we will consider all arise from the following construction.

A category $\mathcal{C}$ is *pre-graded* if there is a function $q : \amalg_{X,Y} \mathrm{Mor}(X,Y) \to \mathbb{Z}$ such that $q(\alpha \circ \beta) = q(\alpha) + q(\beta)$ whenever the composition makes sense. The category $\mathcal{T}_{n,m}$ is an important example. For $T, T' \in \mathcal{T}_{n,m}$ and $\Sigma : T \to T'$, we define $q(\Sigma) = \chi(\Sigma) - (m+n)/2$. If $T_0, T_1, T_2 \in \mathcal{T}_{n,m}$ and $\Sigma_0 : T_0 \to T_1$, $\Sigma_1 : T_1 \to T_2$, then

$$\chi(\Sigma_0 \circ \Sigma_1) = \chi(\Sigma_0) + \chi(\Sigma_1) - \chi(T_1) = \chi(\Sigma_0) + \chi(\Sigma_1) - (n+m)/2$$

by additivity of the Euler characteristic. It follows that $q(\Sigma_0 \circ \Sigma_1) = q(\Sigma_0) + q(\Sigma_1)$.

If $\mathcal{C}$ is pre-graded, we construct a graded category $\mathcal{C}^{gr}$ as follows. Objects of $\mathcal{C}^{gr}$ are pairs (X, n) where X is an object of $\mathcal{C}$ and $n \in \mathbb{Z}$. and we then define

$$\mathrm{Mor}_{\mathcal{C}^{gr}}((X,n),(Y,m)) = \{\alpha \in \mathrm{Mor}_{\mathcal{C}}(X,Y) \,|\, q(\alpha) = n - m\}.$$

The grading shift functor is given on objects by $q((X,n)) = (X, n+1)$ and on morphisms by $q(\alpha) = \alpha$.

Exercise 4.4.1. Check that $\mathcal{C}^{gr}$ is a category and that q is a functor.

If $\mathcal{C}$ is a pre-graded cateory, we will denote the object (X, n) of $\mathcal{C}^{gr}$ by $q^n X$.

Additive Categories: The category $\mathcal{C}$ is *additive* if

1) For all $X, Y \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Mor}(X,Y)$ is a $\mathbb{Z}$-module;
2) composition of morphisms is bilinear; and
3) there is an object $X \oplus Y \in \mathrm{Obj}(\mathcal{C})$ satisfying the usual categorical definition of direct sum.

If $\mathcal{C}$ is an additive category and $X, Y \in \mathrm{Obj}(\mathcal{C})$, we write $\mathrm{Hom}(X,Y)$ in place of $\mathrm{Mor}(X,Y)$. Let $X_1, \ldots, X_r$ be a set of objects of $\mathcal{C}$. If every object of $\mathcal{C}$ is isomorphic to a finite direct sum of the C_i's, we say that $X_1, \ldots, X_r$ *generate* $\mathcal{C}$.

More generally, if $\mathcal{C}$ is graded, we say that $X_1, \ldots, X_r$ generate $\mathcal{C}$ if any object is isomorphic to a finite direct of shifted copies of the X_i and write

$$\mathrm{Hom}^q(X,Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(X, q^i Y).$$

For any category $\mathcal{C}$, we have an additive category $\mathrm{Add}(\mathcal{C})$ with objects given by finite formal direct sums $\bigoplus_{j=1}^n X_n$, $X_j \in \mathrm{Obj}\,\mathcal{C}$, morphisms by matrices of formal linear combinations of morphisms in $\mathcal{C}$, and composition by multiplication of matrices. More precisely, if G_{ij} is the free $\mathbb{Z}$-module generated by $\mathrm{Mor}(X_j, Y_i)$, then we have

$$\mathrm{Hom}(\bigoplus_{j=1}^{n} X_j, \bigoplus_{i=1}^{m} Y_i) = \{m \times n \text{ matrices } [\alpha_{ij}] \,|\, \alpha_{ij} \in G_{ij}\}.$$

Field Coefficients: If $\mathcal{C}$ is an additive category and A is a $\mathbb{Z}$-module, we define the category $\mathcal{C} \otimes A$ to be the category whose objects are the objects of $\mathcal{C}$ and whose morphism spaces are given by $\mathrm{Hom}_{\mathcal{C}\otimes A}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathbb{Z}} A$. A case of particular interest to us is when $A = \mathbb{F}$ is a field.

Quotient Categories: Suppose $\mathcal{C}$ is an additive category. An *ideal* $\mathcal{I}$ in $\mathcal{C}$ consists of a subgroup $\mathcal{I}_{XY} \subset \mathrm{Hom}(X,Y)$ for each X and Y in $\mathrm{Obj}(\mathcal{C})$ such that for every $\alpha \in \mathrm{Hom}(W,X)$, $\beta \in \mathcal{I}_{XY}$, and $\gamma \in \mathrm{Hom}(Y,Z)$, $\alpha \circ \beta \in \mathcal{I}_{WY}$ and $\beta \circ \gamma \in \mathcal{I}_{YZ}$ hold.

If $\mathcal{I}$ is an ideal in $\mathcal{C}$, the quotient category $\mathcal{C}/\mathcal{I}$ is defined to have the same objects as $\mathcal{C}$ and morphisms given by $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)/\mathcal{I}_{XY}$. Note that although the two categories have the same objects, two objects which are not isomorphic in $\mathcal{C}$ may well become isomorphic in $\mathcal{C}'$.

Example 4.4.2. Let $\mathcal{C}$ be the category of chain complexes over a ring R. $\mathcal{C}$ is an additive category. It contains an ideal $\mathcal{J}$ which consists of all chain maps f which are chain homotopic to the 0 map (*i.e.* $f = dh + hd$ for some h.) The quotient $\mathcal{C}/\mathcal{J}$ is the homotopy category of chain complexes over R.

Exercise 4.4.3. Let $\mathcal{P}$ be the category of planar tangles, and let $R = \mathbb{Z}[q^{\pm}1]$. Define $\mathbf{P}$ to be the quotient of $\mathrm{Add}(\mathcal{P}) \otimes R$ by the ideal generated by the local relation $\bigcirc = q + q^{-1}$. Show that $\mathrm{Hom}_{\mathbf{P}}(X_n, X_m) = V_{n,m}$, and that $\cdot : V_{n,m} \times V_{m,l} \to V_{n,l}$ is composition in $\mathbf{P}$.

4.5. The Krull-Schmidt property

Definition 4.5.1. An additive graded category $\mathcal{C}$ is *positively graded* if $\mathcal{C}$ is generated by objects $X_1, X_2, \dots X_r$ which satisfy

(1) $\mathrm{Hom}(X_i, q^n X_j) = 0$ for any $n < 0$ and all i, j.
(2) $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$.
(3) $\mathrm{Hom}(X_i, X_i) \simeq \mathbb{Z}$ is generated by id_{X_i}.

If $\mathcal{C}$ is positively graded, it has a decreasing sequence of ideals $\mathcal{J}_0 \supset \mathcal{J}_1 \supset \dots$ where $\mathcal{J}_n$ is generated by morphisms in $\mathrm{Hom}(X_i, q^m X_j)$ for all i, j and every $m > n$. Note that $\mathrm{Hom}(Y, Y') \cap \mathcal{J}_n = 0$ for all but finitely many n, since Y and Y' can each be written as a finite direct sum of the X_i's.

The quotient $\mathcal{C}/\mathcal{J}_1$ is the *associated graded category*; it satisfies

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{J}_1}(q^a X_i, q^b X_j) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X_i, X_j) & a = b \\ 0 & a \neq b \end{cases}.$$

Proposition 4.5.2. *Suppose* Y *and* Y' *are objects of a positively graded category* $\mathcal{C}$. *If*

$$Y = \bigoplus_{j=1}^{N} q^{a_j} X_{f(j)} \simeq \bigoplus_{k=1}^{M} q^{b_k} X_{g(k)} = Y',$$

then $N = M$ *and there is a bijection* $\varphi : \{1, \dots, N\} \to \{1, \dots, N\}$ *such that* $b_{\varphi(j)} = a_j$ *and* $g(j) = f(\varphi(j))$.

In other words, Y can be uniquely decomposed into a direct sum of shifted copies of X_i. This is known as the *Krull-Schmidt property.*

Proof. First suppose all the a_j's and b_k's are 0. Then $\mathrm{Hom}(Y, Y')$ is a direct sum of algebras $M_{n_i \times m_i}(\mathbb{Z})$, where m_i is the size of the set $f^{-1}(i)$ and n_i is the size of $g^{-1}(i)$. For $\mathrm{Hom}(Y, Y')$ to contain an invertible element, we must have $n_i = m_i$ for all $i = 1, \dots, r$.

For the general case, let $\overline{Y} = \bigoplus_{n \in \mathbb{Z}} q^n \overline{Y}_n$ and $\overline{Y}' = \bigoplus_{n \in \mathbb{Z}} q^n \overline{Y}'_n$ be the images of Y and Y' in $\mathcal{C}/\mathcal{J}_1$. Then $\mathrm{Hom}(\overline{Y}, \overline{Y}') = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\overline{Y}_n, \overline{Y}'_n)$. Since $\overline{Y} \simeq \overline{Y}'$, we see that $\overline{Y}_n \simeq \overline{Y}'_n$ for all n. The lemma now follows from the the first case. □

Corollary 4.5.3. *If* $f \in \mathrm{Hom}(Y, Y')$ *and the associated graded morphism* $\overline{f} : \overline{Y} \to \overline{Y}'$ *is an isomorphism, then* f *is an isomorphism.*

Proof. Write $f = f_0 + f_1$, where f_0 consists of those components of f in summands of the form $\mathrm{Hom}(q^n X_i, q^n X_j)$ and f_1 consists of those components in summands $\mathrm{Hom}(q^n X_i, q^m X_j)$ where $n < m$. Then $f_0 = \overline{f}$ is an isomorphism and $f_1 \in \mathcal{J}_1$. Write $g = -f_0^{-1} f_1$, so $f = f_0(\mathrm{id}_Y - g)$. Then $f^{-1} = \left(\sum_{i=0}^{\infty} g^i\right) f_0^{-1}$. Note that $g^i \in \mathcal{J}_i$, so all but finitely many terms of the sum are 0. □

4.6. Chain complexes over a category

Definition 4.6.1. A chain complex over an additive category $\mathcal{C}$ is a sequence

$$\dots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \dots$$

where each $C_i \in \mathrm{Obj}(\mathcal{C})$, $d_i \in \mathrm{Hom}(C_i, C_{i-1})$, and $d_{i-1} \circ d_i = 0$ for all i.

The usual definitions of chain map, chain homotopy, and chain homotopy equivalence can be easily generalized to complexes over $\mathcal{C}$.

In general, it can be hard to tell whether two chain complexes over an arbitrary category $\mathcal{C}$ are homotopy equivalent. If $\mathcal{C}$ is not abelian, we cannot take the homology of a complex over it. However when $\mathcal{C}$ is positively graded, there is an effective way of addressing this question.

If (C, d) is a chain complex over a positively graded category $\mathcal{C}$, define the *associated graded complex* $(\overline{C}, \overline{d})$ to be the image of (C, d) in the associated graded category $\mathcal{C}/\mathcal{J}_1$.

Definition 4.6.2. Suppose that (C, d) is a chain complex over a positively graded category $\mathcal{C}$, and that $(\overline{C}, \overline{d})$ is the associated graded complex. We say C is *minimal* if $\overline{d} = 0$.

Proposition 4.6.3. *Suppose C and C' are finitely generated minimal chain complexes over a positively graded category $\mathcal{C}$. If C and C' are chain homotopy equivalent, they are isomorphic.*

Proof. Suppose $f : C \to C'$, $g : C' \to C$ are chain maps inducing a chain homotopy equivalence. Then $\overline{f} : \overline{C} \to \overline{C}'$, $\overline{g} : \overline{C}' \to \overline{C}$ induce a chain homotopy equivalence in the associated graded category. Since C and C' are minimal, $\overline{d} = \overline{d}' = 0$, and $\overline{f}$ and $\overline{g}$ are isomorphisms. Since C and C' are finitely generated, Corollary 4.5.3 implies that the map $f_i : C_i \to C'_i$ is an isomorphism for each i. Hence f is an isomorphism of chain complexes. □

Starting with an arbitrary complex C, we can try to find a minimal complex by repeatedly applying the following lemma, whose proof is left as an exercise to the reader.

Lemma 4.6.4 (Cancellation)**.** *If $\iota \in \mathrm{Hom}(A, A)$ is an isomorphism, the complex*

$$\to C_{n+1} \xrightarrow{\begin{bmatrix} d_{n+1} \\ f \end{bmatrix}} C'_n \oplus A \xrightarrow{\begin{bmatrix} \alpha & \beta \\ \gamma & \iota \end{bmatrix}} C'_{n-1} \oplus A \xrightarrow{\begin{bmatrix} d_{n-1} & g \end{bmatrix}} C_{n-2} \to$$

is homotopy equivalent to the complex

$$\to C_{n+1} \xrightarrow{d_{n+1}} C'_n \xrightarrow{\alpha-\beta\iota^{-1}\gamma} C'_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to$$

We describe the operation of passing from the larger complex to the smaller one by saying that we *cancel* the two objects isomorphic to A.

Proposition 4.6.5. *Suppose $\mathcal{C}$ is a positively graded category with coefficients in a field $\mathbb{F}$. Then any finitely generated complex over $\mathcal{C}$ is homotopy equivalent to a minimal complex.*

Proof. By induction on the total number of summand in C. Since $\mathcal{C}$ has coefficients in a field, any nonzero component of d which belongs to a summand of the form $\mathrm{Hom}(q^n X_i, q^n X_i)$ is an isomorphism. If $\overline{d} \neq 0$, we can find such a component of d. Canceling produces a homotopy equivalent complex which has fewer summands. □

In practice, many complexes considered here can be reduced to minimal form by repeated application of Lemma 4.6.4 over $\mathbb{Z}$, with no need to pass to a field.

4.7. The Cube of Resolutions We now turn our attention to the problem of categorifying the Jones polynomial of a tangle.

Definition 4.7.1. Suppose that $D_0, D_1 \in \mathcal{P}_{n,m}$ are planar n-tangles. A *marked cobordism* $\mathcal{S} : D_0 \to D_1$ is a pair $\mathcal{S} = (S, P)$ where $S \hookrightarrow \mathbb{R} \times I \times I$ is a cobordism from D_0 to D_1, and $P \subset \operatorname{int} S$ is a finite set of *marked points*. Marked cobordisms $(S_1, P_1), (S_2, P_2) : D_0 \to D_1$ are said to be equivalent if there is a homeomorphism $\varphi : \mathbb{R} \times I \times I$ which takes (S_1, P_1) to (S_2, P_2) and is the identity on $\partial\mathbb{R} \times I \times I$.

Let $\mathcal{C}$ be the category whose objects are planar (n, m) tangles and whose morphisms are marked cobordisms between them. $\mathcal{C}$ is pre-graded, with

$$q(S, P) = \chi(S) - 2|P| - (m + n)/2.$$

We define $\mathbf{Cob}_{n,m} = \mathcal{C}^{gr}$. Just as for tangles in $\mathbb{R}^2 \times I$, there is a 2-category **Cob** whose 1-morphisms are planar tangles and whose 2-morphisms are marked cobordisms between them. In particular, we have horizontal and vertical compositions of marked tangles whose definitions are completely analogous to those for planar tangles.

Given a (m, n) tangle diagram D, we can form the cube of resolutions of D. We label each vertex v of the cube with its associated resolution $q^{|v|}D_v$, which is an object of $\mathbf{Cob}_{n,m}$. Similarly, we label each edge $e : v_0 \to v_1$ with the morphism $\mathcal{S}_e = (S_e, \emptyset) : q^{|v_0|}D_{v_0} \to q^{|v_1|}D_{v_1}$. Here S_e is the standard saddle cobordism in a neighborhood of the crossing, and the product cobordism outside of it. We have $q(\mathcal{S}_e) = \chi(S_e) - (n + m)/2 = -1 = |v_0| - |v_1|$, so $\mathcal{S}_e$ is a morphism in the graded category $\mathbf{Cob}_{n,m}$.

To build a cochain complex $C(D)$ from this cube, we pass to the additive category $\mathrm{Add}(\mathbf{Cob}_{m,n})$, in which we are allowed to take formal direct sums of objects and formal linear combination of morphisms.

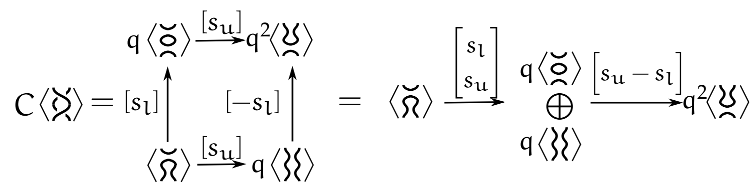


Figure 4.7.2. An example of a complex $C(D)$

Definition 4.7.3. At the level of objects, the complex $C(D)$ is given by
$$C^i(D) = \bigoplus_{|v|=i} q^i D_v,$$

The differential $d_i : C^i(D) \to C^{i+1}(D)$ will be a matrix whose columns are labeled by vertices with $|v| = i$ and whose rows are labeled by vertices with $|v| = i+1$. If $|v_0| = i$ and $e : v_0 \to v_1$, the corresponding entry in the matrix for d_i will be $(-1)^{\sigma(e)}\mathcal{S}_e$, where $\sigma(e)$ is a sign assignment as in Lemma 3.3.1; if there is no such edge, the entry will be the 0 morphism. The fact that $d^2 = 0$ follows exactly as in the closed case.

Example 4.7.4. Figure 4.7.2 shows the complex $C(D)$, where D is the planar diagram at the left of the figure. The morphisms s_u and s_l are handle attachments at the site of the upper and lower crossings.

Composition: Horizontal composition gives a functor $\mathbf{Cob}_{n,m}\times\mathbf{Cob}_{m,l}\to\mathbf{Cob}_{n,l}$. We extend this functor bilinearly over the ring $R = \mathbb{Z}[q^{\pm 1}]$ to get a functor $\mathcal{H} : \mathbf{Cob}_{n,m} \otimes_R \mathbf{Cob}_{m,l} \to \mathbf{Cob}_{n,l}$ which on objects is given by $\mathcal{H}(D \otimes D') = DD'$.

Proposition 4.7.5. $C(DD') = \mathcal{H}(C(D) \otimes C(D'))$.

Proof. At the level of objects, this follows just as in the proof of Proposition 4.3.3. For the differential note that an edge in the cube of resolutions for DD' is either of the form $e : v_0v_0' \to v_1v_0'$ where $e : v_0 \to v_1$ is an edge of the cube of resolutions for D, or of the form $e' : v_0v_0' \to v_0v_1'$, where $e' : v_0' \to v_1'$ is an edge in the cube of resolutions for D'. Edges of the first and second types contribute terms respectively of the form $d \otimes 1$, and $1 \otimes d'$ to the differential. □

4.8. Bar-Natan's category At this point, we have effectively generalized most of the steps in the definition of Khovanov homology to tangles, but the complex we have defined is not invariant (even up to homotopy equivalence) under the Reidemeister moves. It remains to find an appropriate generalization of the final step: applying the TQFT $\mathcal{A}$. It turns out that the right thing to do is to pass to a certain quotient of the category $\mathrm{Add}(\mathbf{Cob}_{n,m})$.

Definition 4.8.1. *Bar-Natan's category* $\mathcal{C}^{BN}_{n,m}$ is the quotient of $\mathrm{Add}(\mathbf{Cob}_{n,m})$ by the ideal generated by the following local relations. Suppose $\mathcal{S} = (S, P)$ is a morphism in $\mathbf{Cob}_{n,m}$.

- (Sphere relations) If a component of S is a sphere with one marked point, $\mathcal{S} = \mathcal{S}'$, where $\mathcal{S}'$ is the cobordism obtained by deleting this component. If some component of S is a sphere with any number of marked points other than one, $\mathcal{S}$=0.
- (Neck-cutting relation) If $A \subset S$ is an annulus, let $\mathcal{S}_r$ (resp. $\mathcal{S}_l$) be the diagram obtained by removing A replacing it with a pair of disks, and putting a marked point on the left-hand (resp. right-hand) disk. Then $\mathcal{S} = \mathcal{S}_r + \mathcal{S}_l$.

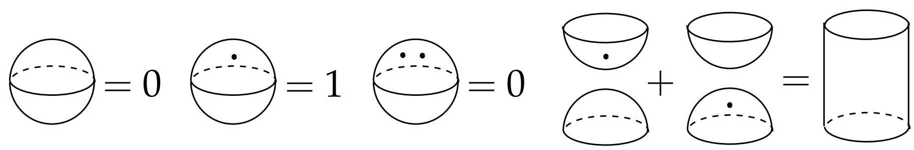


Figure 4.8.2. Relations in $\mathcal{C}^{BN}_{n,m}$

Exercise 4.8.3. Check that these relations respects the q-grading on morphisms.

Definition 4.8.4. Suppose $D \in \mathcal{D}_{n,m}$. We define the *Khovanov bracket* $\mathrm{CKh}(D)$ to be the image of $C(D)$ under the quotient functor $\mathrm{Add}(\mathbf{Cob}_{n,m}) \to \mathcal{C}^{BN}_{n,m}$.

Using the additional relations in $\mathcal{C}^{BN}_{n.m}$, we will prove that the homotopy type of $\mathrm{CKh}(D)$ is invariant under Reidmeister moves. As an example of how the relations are applied we prove

Lemma 4.8.5. *If any component of* S *contains two or more elements of* P*, then* $\mathcal{S} = 0$ *in* $\mathcal{C}^{BN}$.

Proof. We can find an embedded disk in S containing two marked points. Apply the neck cutting relation to the boundary of this disk. Both the resulting cobordisms contain a sphere with more than one marked point. □

At first glance, the relations in Definition 4.8.1 may seem somewhat mysterious. In fact, they are a natural extension of the TQFT $\mathcal{A}$ to the category of tangles. To see this, consider the category $\mathbf{Cob}_{0,0}$, whose objects are closed 1-manifolds embedded in $\mathbb{R} \times I$, and whose morphisms are marked cobordisms between them. Note that any marked cobordism can be written as a composition of ordinary cobordisms (without markings) and product cobordisms in which one of the product cylinders is marked with a single dot.

We define a functor $\overline{\mathcal{A}} : \mathbf{Cob}_{0,0} \to \mathbb{Z}$-mod. $\overline{\mathcal{A}}$ agrees with $\mathcal{A}$ on objects and unmarked morphisms. If $X_i : D \to D$ is a product cobordism marked with a single point on the ith cylinder,

$$\overline{\mathcal{A}}(X_i) : \mathcal{A}(D) \to \mathcal{A}(D)$$

is given by multiplication by $\mathbf{x}$ on the ith tensor factor of $\mathcal{A}(S^1)$.

Exercise 4.8.6. Check that $\overline{\mathcal{A}}$ descends to a functor $\mathcal{C}^{BN}_{0,0} \to \mathbb{Z}$-mod.

We now consider the structure of $\mathcal{C}^{BN}_{n,m}$.

Proposition 4.8.7 (Eliminating Closed Circles)**.** *Suppose* $D \in \mathcal{P}_{n,m}$ *contains a small closed circle* c*, and let* $\widetilde{D}$ *be the same diagram, but with* c *removed. In* $\mathcal{C}^{BN}_{n,m}$*,* D *is isomorphic to* $q\widetilde{D} \oplus q^{-1}\widetilde{D}$*.*

The proposition categorifies the circle-removal relation $\langle D \rangle = (q + q^{-1})\langle \widetilde{D} \rangle$ for the Kauffman bracket.

Proof. We give the proof in the case where $\widetilde{D}$ is empty; the argument in the general case is the same. There are morphisms $\eta : D \to \widetilde{D}$ and $\varepsilon : \widetilde{D} \to D$ given by addition of a 2-handle and 0-handle, respectively. Define morphisms

$$\varphi : D \to q\widetilde{D} \oplus q^{-1}\widetilde{D} \quad \text{and} \quad \psi : q\widetilde{D} \oplus q^{-1}\widetilde{D} \to D$$

by

$$\varphi = \begin{pmatrix} \eta X \\ \eta \end{pmatrix} \qquad \psi = \begin{pmatrix} \varepsilon & X\varepsilon \end{pmatrix}$$

where $X : D \to D$ is the product cobordism with a single marked point. Then

$$\varphi\psi = \begin{pmatrix} \eta X \varepsilon & \eta X^2 \varepsilon \\ \eta\varepsilon & \eta X \varepsilon \end{pmatrix}$$

The cobordism $\eta\varepsilon$ is a closed sphere, so the sphere relations imply that the diagonal entries of this matrix are 1, and the off-diagonal entries are 0. Similarly

$$\psi\varphi = \varepsilon\eta X + X\varepsilon\eta = 1_D$$

by the neck-cutting relation. □

Corollary 4.8.8. $\mathcal{C}^{BN}_{n,m}$ *is generated (as an additive category) by simple planar tangles.*

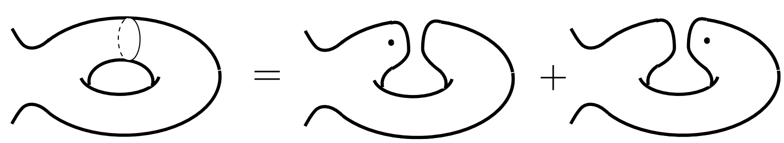


Figure 4.8.9. Trading genus for marked points

Next, we consider the morphisms of $\mathcal{C}^{BN}_{n,m}$. Let $\mathcal{S} : D_0 \to D_1$ be a marked cobordism between simple planar tangles. We say that $\mathcal{S}$ is *simple* if every component of $\mathcal{S}$ is a disk with either 0 or 1 marked point.

Now suppose that $\mathcal{S} : D_0 \to D_1$ is an arbitrary marked cobordism. Applying the neck-cutting relation as shown in Figure 4.8.9, we see that we can reduce the genus of any component of $\mathcal{S}$ at the cost of adding marked points. Similarly, if any component of $\mathcal{S}$ has more than one boundary component, we can use the neck-cutting relation to separate these components. Finally, we can use the sphere relations to eliminate any closed components. What remains is a linear combination of simple cobordisms.

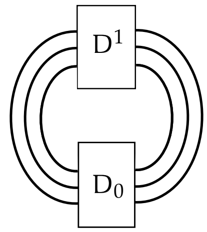


Figure 4.8.10. The closed diagram $\overline{D^1 D_0}$. The planar diagram D^1 is obtained by reflecting D_1 across a horizontal line.

If $\mathcal{S} = (S, P)$ is a simple marked cobordism, the components of S are determined up to isotopy by

$$\partial S = D_0 \times 0 \cup X_n \times 0 \times I \cup D_1 \times 1 \cup X_m \times 1 \times I \subset \partial(\mathbb{R} \times I \times I).$$

We can identify $\partial(\mathbb{R} \times I \times I)$ with $\mathbb{R}^2 - 0$. Under this identification, ∂S is identified with the diagram $\overline{D^1 D_0}$ shown in Figure 4.8.10. In particular, the number of disks is given by the number of components in $\overline{D^1 D_0}$, and each disk can have either 0 or 1 marked point. In summary, we have proved

Proposition 4.8.11. *If D_0 and D_1 are simple planar tangles, $\underline{\mathrm{Hom}}^q(D_0, D_1)$ is a free $\mathbb{Z}$-module of rank 2^r, where r is the number of components of $\overline{D^1 D_0}$.*

The simple marked cobordism with maximal grading is $(S, \emptyset)$, for which we have $q(S, \emptyset) = r - (m+n)/2$. It follows that as a graded module, $\mathrm{Hom}^q(D_0, D_1)$ is naturally isomorphic to $q^{(m+n)/2}\mathcal{A}(\overline{D^1 D_0})$. In particular, $\mathrm{Hom}(D_0, q^i D_1) = 0$ for $i < r - (m+n)/2$. It is easy to see that $r \leqslant (m+n)/2$, with equality if and only if $D_0 = D_1$, and that $\mathrm{Hom}(D_0, D_0) \simeq \mathbb{Z}$ is generated by id_{D_0}. Hence we have proved

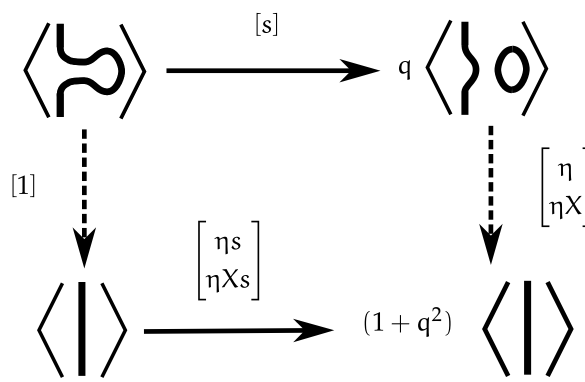
Corollary 4.8.12. *$\mathcal{C}^{BN}_{n,m}$ is positively graded.*

This corollary has two powerful consequences which make computations with $\mathrm{CKh}(D)$ feasible. First, $\mathcal{C}^{BN}_{n,m}$ has the Krull-Schmidt property. Second, by Proposition 4.6.3, any two minimal representatives of $\mathrm{CKh}(D)$ are isomorphic.

4.9. How to Compute Given a tangle diagram D, we can simplify $\mathrm{CKh}(D)$ by the following steps. First, we apply the isomorphism of Proposition 4.8.7 to eliminate any objects of $\mathrm{CKh}(D)$ which contain closed circles. The result is a chain complex whose objects are direct sums of simple tangles. Next, we use the relations in Definition 4.8.1 to express all morphisms in the resulting complex in terms of elementary morphisms, as in the proof of Proposition 4.8.11. Finally, we repeatedly apply the Cancellation Lemma to eliminate any components of d which correspond to the identity morphism.

As an example of this process, we prove that CKh is invariant under the first Reidemeister move. Let R_1 and R_1' be the two diagrams for the Reidemeister I move shown in Figure 1.1.6.

Proposition 4.9.1. $\mathrm{CKh}(R_1) \sim \mathrm{CKh}(R_1')$, *where* $\sim$ *indicates that the two complexes are homotopy equivalent up to an overall shift of the homological and* q*-gradings.*

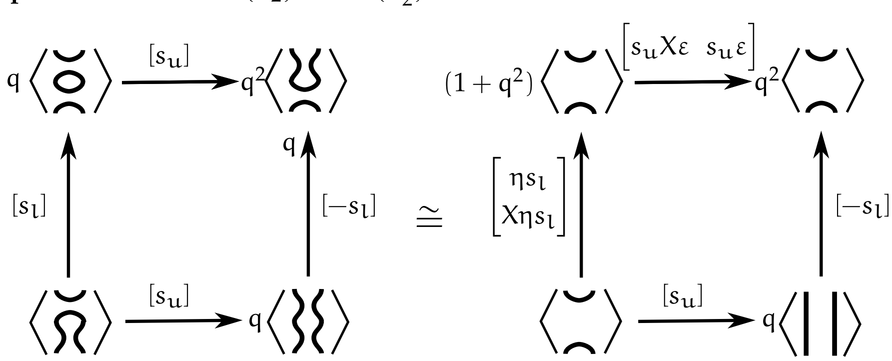


Proof. The complex $\mathrm{CKh}(R_1')$ is shown in the upper row of the figure. The map s is a 1-handle cobordism. The dashed arrows give an isomorphism between the complexes in the upper and lower rows. The morphism ηs is a 1-handle followed by a cancelling 2-handle, so it is isotopic to the identity. Hence we can cancel $\langle | \rangle$ on the left with $\langle | \rangle$ on the right, leaving only a summand of $q^2\langle | \rangle$ in homological grading 1. This is $\mathrm{CKh}(R_1)$ with the homological grading shifted by 1 and the q-grading shifted by q^2. □

Exercise 4.9.2. Show that CKh is invariant under the mirror image of this Reidemeister move.

Invariance under the Reidemeister II move is similar.

Proposition 4.9.3. $\mathrm{CKh}(R_2) \sim \mathrm{CKh}(R_2')$.



Proof. The complex $\mathrm{CKh}(R_2')$ is shown on the left in the figure above. The subscripts s_u and s_l indicate whether the 1-handle should be added to the upper or lower half of the figure. After eliminating the extra circle in the tangle at the upper left, we arrive at the complex on the right-hand side of the figure. The left-hand side of the complex is the same as the complex we considered for the Reidemeister 1 move, so we can cancel the two objects labeled $\langle \asymp \rangle$. Doing this does not affect the morphism on the upper arrow, so we can cancel the two objects labeled $q^2\langle \asymp \rangle$, leaving only $\langle\rangle\langle\rangle$ in the lower-right. $\square$

When we compute CKh of a tangle represented by a braid, it is often simpler to represent tangles by elements of TL_n, rather than drawing pictures. Under this correspondence, we have

$$\mathrm{CKh}(\sigma_1) = 1 \to qU_1 \quad \text{and} \quad \mathrm{CKh}(\sigma_1^{-1}) = U_1 \to q1.$$

Example 4.9.4. Let us compute CKh of a full twist on two strands. Using Proposition 4.7.5, we see that

$$\mathrm{CKh}(\sigma_1)^2 = \mathrm{CKh}(\sigma_1) \otimes \mathrm{CKh}(\sigma_1) = (1 \to qU_1) \otimes (1 \to qU_1)$$

Expanding this out gives

$$\begin{array}{ccc} qU_1 & \xrightarrow{s_u} & q^2U_1^2 \\ {\scriptstyle s_l}\uparrow & & \uparrow{\scriptstyle -s_l} \\ 1 & \xrightarrow{s_u} & qU_1 \end{array} \qquad \begin{array}{ccc} qU_1 & \xrightarrow{1\oplus X_u} & (q+q^3)U_1 \\ {\scriptstyle s_l}\uparrow & & \uparrow{\scriptstyle 1\oplus X_l} \\ 1 & \xrightarrow{s_u} & qU_1 \end{array}$$

where all the maps have already been computed in the proofs of Propositions 4.9.1 and 4.9.3. Cancelling the identity map in the top row, we obtain

$$\mathrm{CKh}(\sigma_1^2) \sim 1 \xrightarrow{s} qU_1 \xrightarrow{X_l - X_u} q^3U_1.$$

Since the saddle cobordism has only one component, $X_u s = X_l s$ and $d^2 = 0$ as it should.

We could do the same thing to compute $\mathrm{CKh}(\sigma_1^3)$, but the following lemma makes the computation a little easier:

Lemma 4.9.5. *If* C, C' *and* C_1 *are complexes and* $C \sim C'$, *then* $C \otimes C_1 \sim C' \otimes C_1$.

The proof is left to the reader. Writing

$$\mathrm{CKh}(\sigma_1^3) = \mathrm{CKh}(\sigma_1^2) \otimes \mathrm{CKh}(\sigma_1)$$

and applying the lemma, we see that $\mathrm{CKh}(\sigma_1^3)$ is equivalent to the complex below:

$$\begin{array}{ccccc} qU_1 & \longrightarrow & (q+q^3)U_1 & \longrightarrow & (q^3+q^5)U_1 \\ \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & qU_1 & \longrightarrow & q^3U_1 \end{array}$$

The complex in the top row gives $\mathrm{CKh}(D)$, where D is U_1 with two Reidemeister one moves applied to it. By R1 invariance, we see that we can cancel all terms in the top row except q^5U_1. The resulting complex is

$$\mathrm{CKh}(\sigma_1^3) \sim 1 \xrightarrow{s} qB_1 \xrightarrow{X_l - X_u} q^3U_1 \xrightarrow{X_l + X_u} q^5U_1.$$

This technique was used to great effect by Bar-Natan and Green in their program [4], which made a revolutionary advance in ability to compute Khovanov homology.

Exercise 4.9.6. Show that in general $\mathrm{CKh}(\sigma_1^n) \sim$

$$1 \xrightarrow{s} qU_1 \xrightarrow{X_l - X_u} q^3U_1 \xrightarrow{X_l + X_u} q^5U_1 \xrightarrow{X_1 - X_2} \ldots q^{2n-3}U_1 \xrightarrow{X_l \pm X_u} q^{2n-1}U_1$$

where the boundary map alternates between $X_l - X_u$ and $X_l + X_u$. By considering one closure of the tangle σ_1^n, compute the Khovanov homology of the torus knot $T(2, 2k+1)$ and the torus link $T(2, 2k)$. Show that the complex obtained by taking the other closure of the tangle correctly computes $\mathrm{Kh}(\bigcirc)$.

We can now check the invariance of CKh under the third Reidemeister move.

Proposition 4.9.7. $\mathrm{CKh}(\sigma_1\sigma_2\sigma_1) = \mathrm{CKh}(\sigma_1\sigma_2\sigma_1)$.

Proof. We compute

$$\begin{aligned} \mathrm{CKh}(\sigma_1\sigma_2) &= (1 \to qU_1) \otimes (1 \to qU_2) \\ &= 1 \to q(U_1 + U_2) \to q^2U_1U_2 \end{aligned}$$

where every matrix entry in each boundary map is a saddle cobordism. Thus $\mathrm{CKh}(\sigma_1\sigma_2\sigma_1)$ has the form shown below:

$$\begin{array}{ccccc} qU_1 & \longrightarrow & (q+q^3)U_1 + q^2U_2U_1 & \longrightarrow & q^3U_1 \\ \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & q(U_1+U_2) & \longrightarrow & q^3U_1U_2 \end{array}$$

Consider the morphisms $F : qU_1 \to qU_1$ and $G : q^3U_1 \to q^3U_1$ appearing in the upper row. The morphism F has already been studied in our calculation of $\mathrm{CKh}(\sigma_1^2)$, where we saw it is the identity. G is given by the composition of the map $\varepsilon : q^3U_1 \to q^2D$, where D is the diagram to the right with the 1-handle cobordism indicated by the dotted line. This is the addition of a 0-handle followed by a cancelling 1-handle, so the composition is isotopic to the identity.

It follows that we can cancel all terms in the upper row except U_1U_2. We are left with a complex

$$1 \to q(U_1 + U_2) \to q^2(U_1U_2 + U_2U_1)$$

in which every possible map is given (up to sign) by a saddle. But this complex is invariant under the reflection which switches U_1 and U_2. Hence the same argument shows it is isomorphic to $\mathrm{CKh}(\sigma_2\sigma_1\sigma_2)$. □

Theorem 4.9.8. *If D and D' two diagrams having the same tangle, then we have* $\mathrm{CKh}(D) \sim \mathrm{CKh}(D')$.

Proof. We've already checked the statement for the three pairs of Reidemeister diagrams. The general case follows exactly as in the proof of Proposition 4.3.4, using Proposition 4.7.5 and Lemma 4.9.5. □

4.10. Connections and Further Reading The Khovanov homology of tangles serves as a template for many subsequent developments in categorification. For the beginner, it is worth taking the time to understand it well; Bar-Natan's paper [5] is still a good reference. One important feature which we have not touched on in this section is maps induced by tangle cobordisms. The proof of (projective) functoriality is much easier in this setting [5, 41] than if we restrict ourself to the context of link cobordisms.

In a different direction, Ozsváth and Szabó [62, 63] have developed a knot Floer analog of the Khovanov complex of a tangle. The underlying categories are more complicated than the Bar-Natan category, and it seems there is still much to be learned by studying them.

Finally, we mention the recent work of Kotelskiy, Watson, and Zibrowius, [47], who represent the Khovanov complex of a tangle as an object in the Fukaya category of the 4-punctured sphere.

5. HOMFLY-PT Homology

The similarity between the skein relations for the Jones and Alexander polynomials is hard to miss, and it's natural to ask if they have a common generalization. In fact, this question is so natural that it was answered by at least eight people working in five different groups (Freyd-Yetter, Hoste-Lickorish, Morton, Ocneanu, and Pryztycki-Traczyk) not long after Jones announced the existence of his new polynomial.

Theorem 5.0.1 (HOMFLY-PT [24] [64]). *There's an invariant which assigns to an oriented link $L \subset S^3$ a rational function $P_L(a, q) \in \mathbb{Z}(a, q)$ which satisfies the skein relation*

$$aP\left(\nwarrow\!\!\!\!\nearrow\right) - a^{-1}P\left(\nwarrow\!\!\!\!\nearrow\right) = (q - q^{-1})P\left(\uparrow\uparrow\right).$$

The function P_L is known as the (normalized) HOMFLY-PT polynomial. It is completely determined by the skein relation plus the normalization $P(\bigcirc) = 1$. By applying the skein relation to the one-crossing diagram of the unknot, we find that

$$P(\bigcirc\,\bigcirc) = \frac{a - a^{-1}}{q - q^{-1}}$$

and more generally, that

$$P(\bigcirc^n) = \left(\frac{a - a^{-1}}{q - q^{-1}}\right)^{n-1}.$$

Notation: We write

$$\{n\} = \frac{aq^{-n} - a^{-1}q^n}{q - q^{-1}} \quad \text{and} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

so that $P(\bigcirc^n) = \{0\}^{n-1}$. Note that $\{n\}|_{a=q^k} = [k - n]$. The polynomial $[n]$ is known as *quantum* n.

As with the Jones polynomial, there's also an *unnormalized* version $\overline{P}_L$ which is defined by

$$\overline{P}(L) = P(L)\left(\frac{a - a^{-1}}{q - q^{-1}}\right).$$

Exercise 5.0.2. Use the skein relation to show that $(q - q^{-1})^{|L|}\overline{P}(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$. In particular, if K is a knot, $P(K) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$.

Our goal in this lecture is to discuss one way of defining P_L (following the method of Jones [36] and Ocneanu, and then outline its categorification, due to Khovanov and Rozansky.

5.1. Braid Closures Given $\sigma \in \mathrm{Br}_n$, we can form the *braid closure* $\overline{\sigma}$, which is an oriented link in S^3, as illustrated in the figure above. By sliding σ "around the circle" as shown, we see that $\overline{\sigma\tau} = \overline{\tau\sigma}$; in other words, that the operation of taking the closure is invariant under conjugation in the braid group.

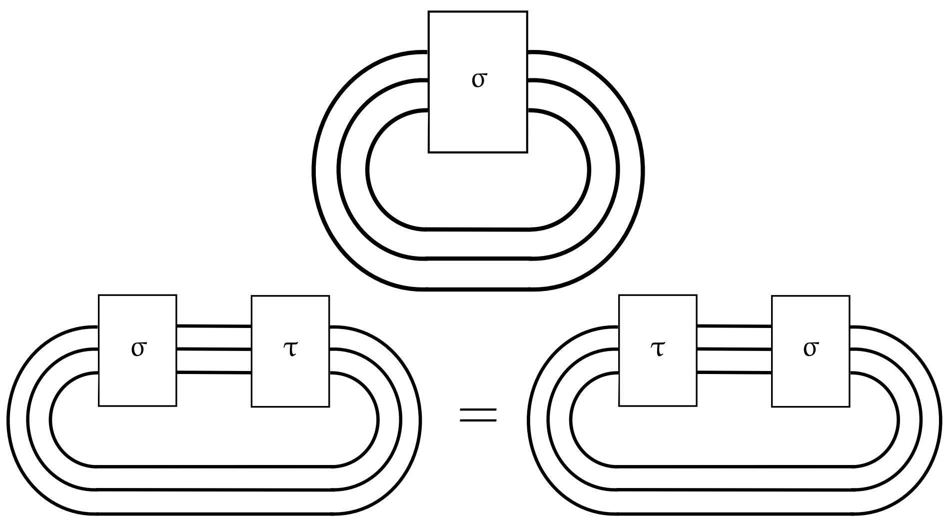


Figure 5.1.1. From the schematic diagram of closure of the braid σ above we see that $\overline{\sigma\tau} = \overline{\tau\sigma}$ below.

The significance of this operation stems from the following two theorems:

Theorem 5.1.2 (Alexander)**.** *Every oriented link* $L \subset S^3$ *can be written as* $L = \overline{\sigma}$ *for some* $\sigma \in \mathrm{Br}_n$.

Note that n depends on L, and that neither n nor σ are uniquely determined by L. This failure of uniqueness is described by Markov's theorem:

Theorem 5.1.3 (Markov)**.** *If* $\sigma \in \mathrm{Br}_n$, $\sigma' \in \mathrm{Br}_{n'}$ *have isotopic closures, then* σ *and* σ' *are related by a sequence of the following moves:*

1) *(Conjugation) Replace* $\sigma \in \mathrm{Br}_m$ *with* $\tau\sigma\tau^{-1}$, *where* $\tau \in \mathrm{Br}_m$.
2) *(Stabilization) Replace* $\sigma \in \mathrm{Br}_m$ *with* $\sigma\sigma_m^{\pm 1} \in \mathrm{Br}_{m+1}$ *or vice-versa.*

Next, we recall the Temperly-Lieb algebra TL_n from the previous lecture. If $U \in \mathsf{TL}_n$ is a basis element corresponding to a crossingless planar diagram, we can form its closure $\overline{U}$ just as we formed the closure of a braid. We define a map $\mathrm{Tr} : \mathsf{TL}_n \to \mathbb{Z}[q^{\pm 1}]$ by setting

$$\mathrm{Tr}(U) = (q + q^{-1})^{|\overline{U}|}$$

if U is a crossingless planar tangle, and extending linearly to all of TL_n. This map is called the *Jones trace*. Since $\overline{UU'} = \overline{U'U}$, it satisfies the defining property of a trace: $\mathrm{Tr}(UU') = \mathrm{Tr}(U'U)$.

Using the homomorphism $\Psi_2 : \mathrm{Br}_m \to \mathsf{TL}_m$ defined in exercise 4.3.7, we see that the unnormalized Jones polynomial can be expressed as

$$\overline{V}(\overline{\sigma}) = \mathrm{Tr}\,\Psi_2(\sigma).$$

5.2. The Hecke algebra

Definition 5.2.1. The Hecke algebra of type A_n is the algebra defined over the ring $R = \mathbb{Z}[q^{\pm 1}]$ by the presentation

$$H_n = \left\langle T_1, \dots T_{n-1} \;\middle|\; \begin{array}{c} T_i T_j = T_j T_i \quad |i-j| > 1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ T_i^2 = (q - q^{-1}) T_i + 1 \end{array} \right\rangle.$$

There are obvious homomorphisms,

$$\mathrm{Br}_n \xrightarrow{\Psi} H_n \xrightarrow{\rho} \mathbb{Z}[S_n]$$

given by $\Psi(\sigma_i) = T_i$ and $\rho(T_i) = s_i$, where S_n is the nth symmetric group, which has a presentation in terms of elementary transpositions s_i for $i = 1, \dots, n-1$ as

$$S_n = \left\langle s_1, \dots s_{n-1} \;\middle|\; \begin{array}{c} s_i s_j = s_j s_i \quad |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i^2 = 1 \end{array} \right\rangle.$$

The presentation of H_n above was chosen to make this sequence of homomorphisms obvious. It is also useful to work with the alternate generators $B_i = q - T_i$, with respect to which the presentation has the form

$$H_n = \left\langle B_1, \dots B_{n-1} \;\middle|\; \begin{array}{c} B_i B_j = B_j B_i \quad |i-j| > 1 \\ B_i B_{i+1} B_i - B_i = B_{i+1} B_i B_{i+1} - B_{i+1} \\ B_i^2 = (q + q^{-1}) B_i \end{array} \right\rangle.$$

With this set of generators, it's clear there is a homomorphism $p_2 : H_n \to \mathsf{TL}_n$ given by $p_2(B_i) = U_i$, and that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Br}_n & & \\ \downarrow{\scriptstyle \Psi} & \searrow^{\Psi_2} & \\ H_n & \xrightarrow{p_2} & \mathsf{TL}_n. \end{array}$$

Here Ψ_2 is basically the map discussed in section 4.3, but with q and q^{-1} switched.

In analogy with the Jones trace, we have the *Jones-Ocneanu trace* on the Hecke algebra:

Theorem 5.2.2 (Oceanu). *n There is a unique family of* R*-linear maps*

$$\mathrm{Tr}_n : H_n \to \mathbb{Z}[a^{\pm 1}, q^{\pm 1}, (q - q^{-1})^{-1}]$$

satisfying the following properties

1) $\mathrm{Tr}_{n+1} BB' = \mathrm{Tr}_{n+1} B'B$
2) $\mathrm{Tr}_{n+1} \iota(B) = \{0\} \mathrm{Tr}_n(B)$
3) $\mathrm{Tr}_{n+1} \iota(B) B_n = \{1\} \mathrm{Tr}_n(B)$
4) $\mathrm{Tr}_0(1) = 1$.

where $\iota : H_n \to H_{n+1}$ *is the inclusion which sends* $B_i \mapsto B_i$.

We can use the Jones-Ocneanu trace to give a definition of the HOMFLY-PT polynomial.

Definition 5.2.3. If $\sigma \in \mathrm{Br}_n$, we define $\overline{P}(\overline{\sigma}) = a^{-w(\sigma)} \mathrm{Tr}_n \Psi(\sigma)$.

By Alexander's theorem, any link in S^3 can be represented as a braid closure. To see that P is well-defined, we use Markov's theorem, which says that it is enough to check that P is invariant under the two Markov moves.

Exercise 5.2.4. Use the properties of the Ocneanu trace to check that $\overline{P}$ is invariant under the Markov moves. Next, use the quadratic relations in H_n to show that if D is any diagram of a braid closure, then $P(D)$ satisfies the HOMFLY-PT skein relation.

5.3. Structure of H_n In this section, we give a proof of Theorem 5.2.2. Along the way, we must establish some facts about the structure of H_n.

Let $\mathcal{W}_n$ be the set of all words in the letters $1, 2, \ldots n-1$. If $I = i_1 i_2 \cdots i_k \in \mathcal{W}_n$, we let $\ell(I) = k$ be its length. We write $s(I) = s_{i_1} s_{i_2} \cdots s_{i_k} \in S_n$ and similarly for $B(I) = B_{i_1} B_{i_2} \cdots B_{i_k} \in H_n$. If $j \leqslant i$, We write $I_{j,i} = j(j+1)(j+2)\cdots(i-1)$; note that $I_{i,i}$ is the empty word. It is easy to see that $s(I_{j,i})(i) = j$.

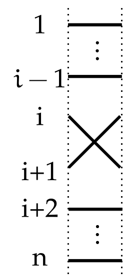
Definition 5.3.1. A *reduced word* is a word of the form $I_{j_n,n} I_{j_{n-1},n-1} \cdots I_{j_1,1}$, where $j_k \leqslant k$ for all $1 \leqslant k \leqslant n$.

Let $\mathcal{R}_n = \{I \in \mathcal{W}_n \mid I \text{ is reduced}\}$.

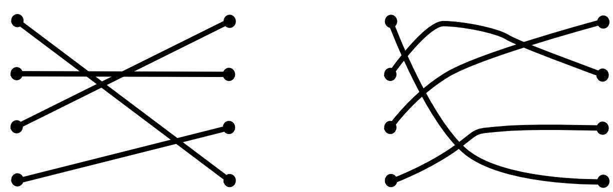
Lemma 5.3.2. *Every* $s \in S_n$ *has a unique expression as* $s = s(I)$, *where* $I \in \mathcal{R}_n$.

Proof. By induction on n. The result for $n = 1$ is obvious. Suppose the lemma holds for $n-1$, and let $s \in S_n$. Let $j = s(n)$. Then $s(I_{j,n})^{-1} s(n) = n$, so we can view $s(I_{j,n})^{-1} s$ as an element of S_{n-1}. By the induction hypothesis we can write $s(I_{j,n})^{-1} s = s(I')$, where I' is a reduced word in the letters $1, \ldots n-2$. Then $s = s(I_{j,n} I')$, and $I_{j,n} I'$ is a reduced word in $n-1$ letters. For uniqueness, note that if $s = s(I)$, where I is a reduced word, we must have $j_k = s(k)$. □

This lemma has an easy graphical interpretation. Any $s \in S_n$ can be represented by a *string diagram*, which is a braid diagram in which we have forgotten the information about overcrossings and undercrossings, and the different strands are allowed to slide freely through each other. The string diagram associated to the elementary permutation s_i is shown in the figure below.



Given a string diagram for a permutation s, suppose the strand that ends at n starts at position i_n. We take this strand and pull it down and to the right, so that the initial part of the word associated to the new diagram reads $i_n i_{n+1} \dots n-1$. The rest of the diagram consists of a diagram on $n-1$ strings together with a single string at the bottom. Repeat this process gives the desired reduced word. For example, the figure below shows that we can write the permutation given in cycle notation by $(134)(2)$ as $s_1 s_2 s_3 s_1$.



Next, we prove an analogous theorem for H_n. Let

$$\mathcal{B}_{n,k} = \text{span}\,\{B(I) \,|\, I \in \mathcal{R}_n \text{ and } \ell(I) \leqslant k\} \subset H_n.$$

Lemma 5.3.3. *If* $I \in \mathcal{W}_n$*,* $B(I) \in \mathcal{B}_{n,\ell(I)}$*.*

Proof. The proof is a double induction. First, we induct on n. The case $n = 1$ is trivial, so suppose the statement holds for $1, \dots, n-1$. We show that it holds for n by induction on $\ell(I)$. Again, the case where $\ell(I) = 0$ is trivial, so suppose the statement holds whenever $\ell(I) < k$. If $\ell(I) = k$, write $I = I' i_k$. By the induction hypothesis, $B(I') \in \mathcal{B}_{n,k-1}$, so it suffices to show that if I is a reduced word of length $k-1$, then $B(I)B_{i_k} \in \mathcal{B}_{n,k}$.

Now suppose that I is a reduced word of length $k-1$ and write $I = I_{i_n,n} I_1$, where $I_1 \in \mathcal{R}_{n-1}$. If $i_k < n-1$, then by the induction hypothesis

$$B(I_1)B_{i_k} = \sum_{J \in \mathcal{R}_{n-1}, \ell(J) \leqslant \ell(I_1 i_k)} c_J B(J).$$

Now if $J \in \mathcal{R}_{n-1}$, $I_{i_n,n}J \in \mathcal{R}_n$, and if $\ell(J) \leqslant \ell(I_1 i_k)$, then $\ell(I_{i_n,n}J) \leqslant \ell(I i_k)$. Hence

$$B(I)B_{i_k} = \sum_J c_J B(I_{i_n,n}J) \in \mathcal{B}_{n,k}.$$

Thus it remains to consider the case where $i_k = n-1$. In this case, write $I = I_{i_n,n} I_{i_{n-1},n-1} I''$, where $I'' \in \mathcal{R}_{n-2}$. If I'' is nonempty, we can commute $i_k = n-1$ with the last letter of I'' to write $B(I)B_{i_k} = B(\tilde{I})B_j$ where $j < n-1$. By the inner induction hypothesis, $B(\tilde{I}) \in \mathcal{B}_{n,k-1}$, so we have reduced to the case of the previous paragraph.

We now suppose that the word I'' is empty. In this case $B(I)$ can be written in one of the following forms: a) $B(I) = B(I_{i_{n-1},n-1})$, b) $B(I) = B(I_{i_n,n})$, or c) $B(I) = B(I_0)B_{n-1}B_{n-2}$, where $I_0 \in \mathcal{W}_{n-2}$. In case a), $B(I)B_{n-1} = B(I_{i_{n-1},n})$, and in case b) $B(I)B_{n-1} = (q+q)^{-1}B(I)$. Finally, in case c)

$$B(I)B_{n-1} = B(I_0)B_{n-2}B_{n-1}B_{n-2} - B(I_0)B_{n-2} + B(I_0)B_{n-1}.$$

The first term on the right is in $\mathcal{B}_{n,k}$ by the argument in the second paragraph, and the other two terms are in $\mathcal{B}_{n,k}$ by the induction hypothesis. This proves the claim. □

Proposition 5.3.4. *The set* $\mathcal{B}_n = \{B(I) \mid I \in \mathcal{R}_n\}$ *is a basis for* H_n.

Proof. By Lemma 5.3.2, $\mathcal{B}_n$ contains exactly $|S_n| = n!$ elements, and it spans H_n by Lemma 5.3.3. On other hand, there is a surjective homomorphism from, H_n to $\mathbb{Z}[S_n]$, so any basis for H_n must contain at least $n!$ elements. □

Proof of Theorem 5.2.2. Suppose Tr_n has been defined; we will inductively construct Tr_{n+1}. We first define Tr_{n+1} on the elements of the basis $\mathcal{B}_{n+1}$; by linearity, this determines Tr_{n+1} on all of H_{n+1}. Now every element of $\mathcal{B}_{n+1}$ is either a) of the form $a \in H_n$ or b) of the form aB_nb, where $a, b \in H_n$. (For simplicity, we have dropped ι from the notation.) In order to satisfy properties 1)-3), we must have $\mathrm{Tr}_{n+1} a = \{0\} \mathrm{Tr}_n a$ and $\mathrm{Tr}_{n+1} aB_nb = \{1\} \mathrm{Tr}_n(ab)$. Hence Tr_{n+1} is uniquely determined.

We define Tr_{n+1} on $\mathcal{B}_{n+1}$ by these relations and extend linearly to H_{n+1}. With this definition, it is easy to see that $\mathrm{Tr}_{n+1} a = \{0\} \mathrm{Tr}_n a$ whenever $a \in H_n$, and $\mathrm{Tr}_{n+1} aB_nb = \{1\} \mathrm{Tr}_n ab$ whenever $a, b \in H_n$. It therefore remains to check that $\mathrm{Tr}_{n+1} xy = \mathrm{Tr}_{n+1} yx$ for $x, y \in H_{n+1}$. To do this, it is enough to check that $\mathrm{Tr}_{n+1} xB_i = \mathrm{Tr}_{n+1} B_ix$ for all $x \in \mathcal{B}_{n+1}$. For $i < n$, this follows from the induction hypothesis.

It remains to prove that $\mathrm{Tr}_{n+1} xB_n = \mathrm{Tr}_{n+1} B_nx$. If $x \in H_n$, this follows from the definition. Otherwise, we can write $x = aB_nb$, for $B, B' \in H_n$. We must check that $\mathrm{Tr}_{n+1} B_naB_nb = \mathrm{Tr}_{n+1} aB_nbB_n$. We consider four cases:

(1) $a, b \in H_{n-1}$.
(2) $a \in H_{n-1}$, $b = b_1B_{n-1}b_2$ with $b_1, b_2 \in H_{n-1}$.
(3) $a = a_1B_{n-1}a_2$ with $a_1, a_2 \in H_{n-1}$, $b \in H_{n-1}$.
(4) $a = a_1B_{n-1}a_2, b = b_1B_{n-1}b_2$ with $a_1, a_2, b_1, b_2 \in H_{n-1}$.

In case (1), a and b commute with B_n, so the result is obvious. In case (2), we compute

$$\mathrm{Tr}_{n+1} B_n a B_n b = [2]\{1\} \mathrm{Tr}_n ab = [2]\{1\}^2 \mathrm{Tr}_{n-1} ab_1b_2$$

while

$$\begin{aligned}\mathrm{Tr}_{n+1} aB_nbB_n &= \mathrm{Tr}_{n+1} aB_nb_1B_{n-1}b_2B_n \\ &= \mathrm{Tr}_{n+1} ab_1B_nB_{n-1}B_nb_2 \\ &= \mathrm{Tr}_{n+1} ab_1(B_{n-1}B_nB_{n-1} - B_{n-1} + B_n)b_2 \\ &= ([2]\{1\}^2 - \{0\}\{1\} + \{1\}\{0\}) \mathrm{Tr}_{n-1} ab_1b_2 \\ &= 2]\{1\}^2 \mathrm{Tr}_{n-1} ab_1b_2\end{aligned}$$

so the result holds. Case (3) is similar. Finally, for case 4, we compute

$$\begin{aligned}\mathrm{Tr}_{n+1} aB_nbB_n &= \mathrm{Tr}_{n+1} ab_1B_nB_{n-1}B_nb_2 \\ &= \mathrm{Tr}_{n+1} ab_1(B_{n-1}B_nB_{n-1} - B_{n-1} + B_n)b_2 \\ &= ([2]\{1\} - 1) \mathrm{Tr}_n ab + \{1\}^2 \mathrm{Tr}_{n-1} a_1a_2b_1b_2\end{aligned}$$

A similar computation shows

$$\mathrm{Tr}_{n+1} B_naB_nb = ([2]\{1\} - 1) \mathrm{Tr}_n ab + \{1\}^2 \mathrm{Tr}_{n-1} a_1a_2b_1b_2$$

which concludes the proof. □

5.4. The cube of resolutions The definition of the HOMFLY-PT polynomial we have given in the preceding sections is modeled on the definition of the Jones polynomial via the Temperly-Lieb algebra. We can rephrase this definition to more closely resemble the Kauffman state model. From section 5.2, we see that

(5.4.1) $$\Psi(\sigma_i) = q - B_i \qquad \Psi(\sigma_i^{-1}) = q^{-1} - B_i.$$

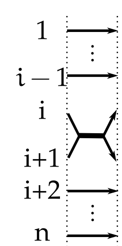
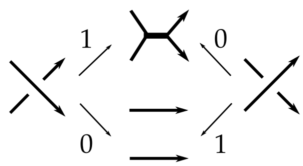


Figure 5.4.2. B_i is represented by a thick edge.

Graphically, we represent B_i by a diagram with a thick edge, as shown in Figure 5.4.2. The resulting diagrams are known as *MOY diagrams*, and were first introduced by Murakami, Ohtsuki, and Yamada.

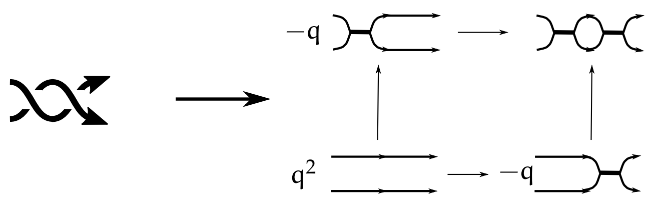
Suppose that c is a crossing in a braid diagram D corresponding to an appearance of $\sigma_i^{\pm 1}$ in the braid word. We consider two ways to "resolve" c: the oriented resolution (corresponding to $1 \in H_n$ and the "thick edge" corresponding to B_i.

We call one of these the 0-resolution, and the other the 1-resolution. Which is which depends on the sign of the crossing, as shown in the figure below.



(**Warning:** if we replace the thick edge by the unoriented resolution we get a convention for 0 and 1 resolutions which is exactly opposite of the one we used in Khovanov homology.)

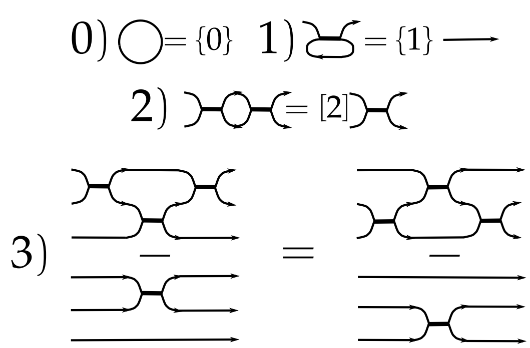
Just as in Khovanov homology, we can form the cube of resolutions of D, as illustrated below.



To each vertex v of the cube we associate a planar MOY diagram D_v, which we view as representing an element $D_v \in H_n$. Using equation 5.4.1, we see that

$$\Psi(D) = q^{n_+(D)} \sum_v (-q)^{-|v|} D_v.$$

If D is an n-strand MOY diagram, we can form its closure $\overline{D}$ exactly as we formed the closure of a braid. In analogy with the Kauffman bracket, we define $\langle \overline{D} \rangle = \mathrm{Tr}_n\, D$. We can evaluate the trace by applying the *MOY relations* shown in the next figure:



MOY relations 0) and 1) are properties 2) and 3) of the trace, while relations 2) and 3) are the defining relations of the Hecke algebra. For example, we have

$$\langle \overline{B_1^2} \rangle = [2]\langle \overline{B_1} \rangle = [2]\{1\}\langle \bigcirc \rangle = [2]\{1\}\{0\}.$$

Finally, comparing with Definition 5.2.3, we see that

$$P(\overline{D}) = a^{-w(D)} q^{n_+(D)} \sum_v (-q)^{-|v|} \langle D_v \rangle.$$

5.5. The Kazhdan-Lusztig basis We'll use the cube of resolutions in the previous section to categorify the HOMFLY-PT polynomial. Before we do so, we must discuss one more thing about the Hecke algebra.

The algebra H_2 has a (rather trivial) filtration by ideals: we have $B_1^2 = [2]B_1$, so $(B_1) \subset (1) = H_2$. More interestingly, a similar phenomenon holds for H_3. Define

$$B_w = B_1B_2B_1 - B_1 = B_2B_1B_2 - B_2.$$

If we use the basis

$$\{1, B_1, B_2, B_{12} := B_1B_2, B_{21} := B_2B_1, B_w\}$$

for the algebra H_3, we see that all the structure constants for multiplication have positive coefficients and there is an increasing sequence of ideals

$$(B_w) \subset (B_1, B_2) \subset (1).$$

Kazhdan and Lusztig [38] defined a similar basis for every H_n, and conjectured that the structure constants defining the multiplication in this basis were all positive. This conjecture was first proved by Beilinson-Bernstein and Brylinski-Kashiwara, and the basis is now known as the Kazhdan-Lusztig basis. We summarise some relevant facts about it in the following:

Theorem 5.5.1 ([7,11,38]). H_n *has a basis* $\{B_s \mid s \in S_n\}$ *such that*

1) $B_sB_t = \sum_u C^u_{st}B_u$, *where* $C^u_{st} \in \mathbb{N}[q^{\pm 1}]$.
2) $\{B_s \mid \ell(s) \leqslant k\}$ *is a basis for* $\mathcal{B}_{n,k}$.
3) *There is a natural map* $|\cdot| : S_n \to \Pi(n)$ *(the set of partitions of* n*) such that the set* $I_\lambda = \langle B_s \mid |s| \succeq \lambda\rangle$ *is an ideal in* H_n.

The partial order $\succeq$ on $\Pi(n)$ is the usual one: $\lambda \succeq \mu$ if the for all k the sum of the k largest elements of λ is $\geqslant$ the sum of the k largest elements of μ. The map $S_n \to \Pi_n$ is much used in the representation theory of S_n, *e.g.* in the Robinson-Schensted correspondence [8].

Exercise 5.5.2. Use the MOY rules to compute Tr_3 of each of the Kazhdan-Lusztig basis elements for H_3. You should be able to express the answers nicely in terms of $\{i\}$ for various values of i. Can you guess where the pattern you see in the answers comes from?

5.6. Soergel Bimodules The proof of Theorem 5.5.1 is one of the first appearances of categorification in representation theory. It proceeds by finding a graded monoidal category $\mathcal{C}_n$ which is generated by objects $\{\mathbb{B}_s \mid s \in S_n\}$ which satisfy

$$\mathbb{B}_s \otimes \mathbb{B}_t \simeq \bigoplus_u C^u_{s,t}\mathbb{B}_s.$$

Since this is the case, we must have $C^u_{s,t} \in \mathbb{N}[q^{\pm 1}]$. There are many different ways to descibe such a category, but the simplest is due to Soergel [75]. The category which he constructed is known as the category of Soergel bimodules, and is denoted by $SBim_n$. This category will play the same role in our definition of the HOMFLY-PT complex of a braid as Bar-Natan's category in the definition of the Khovanov complex of a tangle.

The objects of SBim_n are graded R_n-R_n bimodules over the graded $\mathbb{C}$-algebra $R_n = \mathbb{C}[X_1, \ldots, X_n]$. (Saying that $\mathbb{B}$ is a bimodule over a $\mathbb{C}$-algebra means that $\mathbb{B}$ is vector space over $\mathbb{C}$, and the $\mathbb{C}$ action on both left and right is given by scalar multiplication.) The grading on R_n is given by $q(X_i) = 2$ for all i, and the monoidal structure on SBim_n is given by tensor product of bimodules.

Since R_n is commutative, any bimodule over R_n can be thought of as a module over the larger polynomial ring $R_n^e := R_n \otimes_{\mathbb{C}} R_n \simeq \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots Y_n]$, where the X_i's act by multiplication by X_i on the left, and the Y_i's as multiplication by X_i on the right.

Example 5.6.1. R_n is a bimodule over itself, where the left action of R_n is given by multiplication on the left, and the right action is given by multiplication on the right. (Of course, R_n is commutative, so which side we multiply on doesn't actually matter.) As a module over R_n^e, $R_n = R_n^e/(X_1 - Y_1, \ldots X_n - Y_n)$.

The symmetric group S_n acts on R_n by permuting the X_i's. Let $R_n^{s_i} \subset R_n$ be the ring of elements which are invariant under the elementary permutation s_i. $R_n^{s_i}$ is itself a polynomial ring:

$$R_n^{s_i} \simeq \mathbb{C}[X_1, \ldots, X_{i-1}, e_1^i, e_2^i, X_{i+2}, \ldots, X_n].$$

where $e_1^i = X_i + X_{i+1}$ and $e_2^i = X_i X_{i+1}$ are the elementary symmetric functions in X_i and X_{i+1}. Viewed as a module over $R_n^{s_i}$, R_n is free of rank 2, with basis $\{1, X_i\}$. Hence $R_n \simeq (1+q^2)R_n^{s_i}$ as graded $R_n^{s_i}$-modules. As usual, the notation $q^2 R_n^{s_i}$ indicates a grading shift: $1 \in R_n^{s_i}$ has grading 0, but the same element in $q^2 R_n^{s_i}$ has grading 2.

Definition 5.6.2. The elementary Soergel bimodule $\mathbb{B}_i := \mathbb{B}_{s_i}$ is defined to be

$$\mathbb{B}_i = q^{-1} R_n \otimes_{R_n^{s_i}} R_n.$$

Viewed as a module over R_n^e, we have

$$\begin{aligned}
\mathbb{B}_i &= q^{-1} R_n^e/(e_1^i(X) - e_1^i(Y), e_2^i(X) - e_2^i(Y)) \\
&= q^{-1} R_n^e/((X_i + X_{i+1}) - (Y_i + Y_{i+1}), X_i X_{i+1} - Y_i Y_{i+1}) \\
&= q^{-1} R_n^e/(X_i + X_{i+1} - Y_i - Y_{i+1}, (Y_i - X_i)(Y_i - X_{i+1}))
\end{aligned}$$

The $\mathbb{B}_i$'s satisfy the relations given by the Hecke algebra.

Proposition 5.6.3. $\mathbb{B}_i^2 \simeq (q + q^{-1})\mathbb{B}_i$ *as* R_n-R_n *bimodules.*

Proof. We prove this for $\mathbb{B}_1 \in \mathrm{SBim}_2$; the general case is completely analogous. Write $R = R_2$, $R' = R_2^{s^1}$. We have

$$\begin{aligned}
\mathbb{B}_1 \otimes \mathbb{B}_1 &\simeq q^{-2}(R \otimes_{R'} R) \otimes_R (R \otimes_{R'} R) \\
&\simeq q^{-2} R \otimes_{R'} R \otimes_{R'} R \\
&\simeq q^{-2} R \otimes_{R'} (1+q^2) R' \otimes_{R'} R
\end{aligned}$$

$$\simeq (q^{-1}+q)q^{-1}R \otimes_{R'} R \simeq (q^{-1}+q)\mathbb{B}_1 \qquad \square$$

This proof is algebraically slick, but can be unilluminating if you've never seen something like it before. A more hands-on approach is to work with modules over R_n^e. Suppose $\mathbb{B}, \mathbb{B}'$ are R_n-R_n bimodules, and that $\mathbb{B} \simeq \mathbb{C}[\mathbf{X},\mathbf{Y}]/(f_i(\mathbf{X},\mathbf{Y}))$, $\mathbb{B}' \simeq \mathbb{C}[\mathbf{Y},\mathbf{Z}]/(g_j(\mathbf{Y},\mathbf{Z}))$. Then

$$\mathbb{B} \otimes \mathbb{B}' \simeq \mathbb{C}[\mathbf{X},\mathbf{Y},\mathbf{Z}]/(f_i(\mathbf{X},\mathbf{Y}), g_j(\mathbf{Y},\mathbf{Z})),$$

which we view as a module over $\mathbb{C}[\mathbf{X},\mathbf{Z}] = R_n^e$.

Example 5.6.4. From this perspective, we have

$$\mathbb{B}_1^2 = q^{-2}\mathbb{C}[X_1,X_2,Y_1,Y_2,Z_1,Z_2]/(\mathcal{R})$$

viewed as a module over $\mathbb{C}[X_1,X_2,Z_1,Z_2]$, where the ideal of relations $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} &= (e_j(\mathbf{X}) - e_j(\mathbf{Y}), e_j(\mathbf{Y}) - e_j(\mathbf{Z}))(j=1,2)\\ &= (e_j(\mathbf{X}) - e_j(\mathbf{Y}), e_j(\mathbf{X}) - e_j(\mathbf{Z}))\end{aligned}$$

viewed as a module over $\mathbb{C}[X_1,X_2,Z_1,Z_2]$. The relations $X_1+X_2 = Y_1+Y_2$ and $X_1X_2 = Y_1Y_2$ can be rewritten as $Y_2 = X_1+X_2-Y_1$ and $Y_1^2 = (X_1+X_2)Y_1 - X_1X_2$. We use these relations to eliminate Y_2 and Y_1^2, so

$$\mathbb{B}_1^2 = q^{-2}(\mathbb{B} \oplus Y_1\mathbb{B}) = (q+q^{-1})q^{-1}\mathbb{B}$$

where

$$\begin{aligned}\mathbb{B} &= \mathbb{C}[\mathbf{X},\mathbf{Z}]/(e_j(\mathbf{X}) - e_j(\mathbf{Z}))\ (j=1,2)\\ &= q\mathbb{B}_1.\end{aligned}$$

The relation in Proposition 5.6.3 corresponds to the quadratic relation in the Hecke algebra. It is easy to see that the analog of the far-commutativity relation is satisified: $\mathbb{B}_i\mathbb{B}_j \simeq \mathbb{B}_j\mathbb{B}_i$ if $|i-j|>1$. To prove the analog of the braid relation, we define a bimodule $\mathbb{B}_w$ over R_3 by $\mathbb{B}_w = q^{-3}R_3 \otimes_{R_3^{S_3}} R_3$, where $R_3^{S_3}$ denotes the ring of invariants under the action of S_3 on R_3 by permuting the coordinates. Since $R_3^{S_3}$ is a polynomial ring in the elementary symmetric functions e_1, e_2, e_3, we have

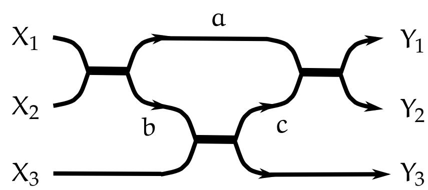
$$\mathbb{B}_w = q^{-3}R_3^e/(e_j(\mathbf{X}) = e_j(\mathbf{Y})) \quad (j=1,2,3).$$

Proposition 5.6.5. $\mathbb{B}_1\mathbb{B}_2\mathbb{B}_1 \simeq \mathbb{B}_w \oplus \mathbb{B}_1$ *as* R_3-R_3 *bimodules.*

Proof. (Sketch) $\mathbb{B}_1\mathbb{B}_2\mathbb{B}_1 = R_3^e[a,b,c]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the relations

$$\begin{aligned} X_1+X_2 &= a+b & X_1X_2 &= ab\\ b+X_3 &= c+Y_3 & bX_3 &= cY_3\\ a+c &= Y_1+Y_2 & ac &= Y_1Y_2\end{aligned}$$

where the variables corresponding to different edges are shown in the figure below.



We can use the linear relations to eliminate a and c, and one of the quadratic relations to eliminate b^2. View $\mathbb{B}_1\mathbb{B}_2\mathbb{B}_1$ as a module over R_3^e, and let M_1 and M_2 be the submodules generated by 1 and $b - X_3$, respectively. Then as a module over R_3^e, $\mathbb{B}_1\mathbb{B}_2\mathbb{B}_1 \simeq M_1 \oplus M_2$. It is easy to check that the maps $R_3^e \to M_1$ and $R_3^e \to M_2$ given by $p \mapsto p$ and $p \mapsto (b - X_3)p$ factor through $\mathbb{B}_w$ and $\mathbb{B}_1$ respectively. Finally, we note that as modules over R_3, $\mathbb{B}_1\mathbb{B}_2\mathbb{B}_1$ is free of rank 8, $\mathbb{B}_w$ is free of rank 6, and $\mathbb{B}_1$ is free of rank 2. It follows that we must have $M_1 \simeq \mathbb{B}_w$ and $M_2 \simeq \mathbb{B}_1$. □

Definition 5.6.6. A *Bott-Samuelson bimodule* is a R_n-R_n bimodule obtained by taking tensor products of the $\mathbb{B}_i$'s. A *Soergel bimodule* is a direct summand of a Bott-Samuelson bimodule.

The category SBim_n is the subcategory of the category of R_n-R_n bimodules whose objects are Soergel bimodules. Its split Grothendieck group $K(\mathrm{SBim}_n)$ is the $\mathbb{Z}[q^{\pm 1}]$-module generated by objects of SBim_n, modulo relations of the form $[\mathbb{B} \oplus \mathbb{B}'] = [\mathbb{B}] + [\mathbb{B}']$ and $[q\mathbb{B}] = q[\mathbb{B}]$. Tensor product on SBim_n makes $K(\mathrm{SBim}_n)$ into a ring: $[\mathbb{B}][\mathbb{B}'] = [\mathbb{B} \otimes \mathbb{B}']$. Propositions 5.6.3 and 5.6.5 imply that there is a homomorphism $\rho : H_n \to K(\mathrm{SBim}_n)$ given by $\rho(B_i) = [\mathbb{B}_i]$. Soergel showed that this map is an isomorphism.

Theorem 5.6.7 (Soergel [75])**.** *There are indecomposable Soergel bimodules* $\mathbb{B}_s$ $(s \in S_n)$ *which generate* SBim_n *and satisfy* $\rho(B_s) = [\mathbb{B}_s]$.

Exercise 5.6.8. Let $\mathbb{B}_{w_n}$ be the $R_n - R_n$ bimodule defined by

$$\mathbb{B}_{w_n} = q^{-n(n-1)/2} R_n \otimes_{R_n^{S_n}} R_n.$$

Show that $\mathbb{B}_i\mathbb{B}_{w_n} = [2]\mathbb{B}_{w_n}$. What is $\mathbb{B}_{w_n}\mathbb{B}_{w_n}$?

Importantly for us, SBim_n satisfies the analog of Corollary 4.8.12.

Theorem 5.6.9 ([75])**.** SBim_n *is a positively graded category;* $\{\mathbb{B}_s \mid s \in S_n\}$ *is a set of positive generators.*

5.7. Hochschild homology and cohomology

Definition 5.7.1. Suppose that R is a $\mathbb{C}$-algebra. If M is an R-R bimodule, its *Hochschild homology* and cohomology are defined as $HH_*(M) = \mathrm{Tor}_*^{R^e}(R, M)$ and $HH^*(M) = \mathrm{Ext}^*_{R^e}(R, M)$.

Here Tor, as usual, denotes the derived tensor product:

$$\mathrm{Tor}_i^{R^e}(R, M) = H_i(C_R \otimes M) = H_i(R \otimes C_M)$$

where C_R and C_M are free resolutions of R and M as R^e modules. Similarly, Ext denotes derived Hom. If R is a graded ring and M is a graded bimodule, then $HH_*(M)$ and $HH^*(M)$ are graded as well.

A priori, $HH_*(M)$ should be a module over R^e, or equivalently, an R-R bimodule. However the left and right actions of R on R are the same, which implies that the same is true for the left and right actions of R on $HH_*(M)$ and $HH^*(M)$, so we can view them as R modules. Hence we have functors

$$HH_*, HH^* : R\text{-gMod-}R \to R\text{-gMod},$$

where R-gMod-R denotes the category of graded R-R bimodules. Note that HH_* and HH^* are both *covariant* functors of M.

When $R = R_n$ is a polynomial ring, the situation simplifies considerably, since R has a very simple resolution as an R^e-module. Namely, if $\mathcal{K}(X_i - Y_i)$ is the short complex $q^2R^e \xrightarrow{X_i - Y_i} R^e$, then the *Koszul complex*

$$C_{R_n} = \bigotimes_{i=1}^{n} \mathcal{K}(X_i - Y_i)$$

is a free resolution of R as a module over R^e.

Example 5.7.2. $HH_*(R_n) = H_*(C_{R_n} \otimes R_n)$. Since $X_i = Y_i$ in R_n,

$$C_{R_n} \otimes R_n \simeq (q^2R_n \xrightarrow{0} \underline{R_n})^{\otimes n}$$

where the underlined terms are in homological grading 0, and the differential lowers the homological grading by 1. The differential in this complex is trivial, so $HH_*(R_n) = R_n \otimes \Lambda^*(a_1, \ldots, a_n)$, where $q(a_i) = 2$ and the homological grading is given by the degree in the exterior algebra.

To compute the Hochschild cohomology, we tensor with the dual chain complex $C^*_{R_n}$. We have

$$C^*_{R_n} \otimes R_n \simeq (\underline{R_n} \xrightarrow{0} q^{-2}R_n)^{\otimes n}$$

where the underlined terms are in homological grading 0 and the differential raises grading by 1. Again, the cohomology is an exterior algebra tensored with R_n.

Example 5.7.3. Let $n = 2$. To compute $HH^*(B_1)$, we use a free resolution of $\mathbb{B}_1$, which is given by the Koszul complex

$$C(\mathbb{B}_1) = (q^2R_2^e \xrightarrow{Y_1+Y_2-X_1-X_2} R_2^e) \otimes (q^4R_2^e \xrightarrow{Y_1Y_2-X_1X_2} R_2^e).$$

Then $HH^*(\mathbb{B}_1) = \mathrm{Hom}(R_n, C(\mathbb{B}_1)) = R_n \otimes \Lambda^*(a_1, a_2)$, but now we have $q(a_1) = 2$, $q(a_2) = 4$.

Proposition 5.7.4. *If* M *and* N *are* $R_n - R_n$*-bimodules, then*

(1) $HH_*(M \otimes N) \simeq HH_*(N \otimes M)$

(2) $\mathrm{Hom}(N, M) \simeq HH^0(M \otimes N^{op})$ *where* N^{op} *denotes* N *with the right and left actions reversed.*

Proof. We prove the first part; the second follows from standard properties of Ext. Let C_M and C_N be free resolutions of M and N. We view them as being defined over polynomial rings with variables X, Y and X', Y', respectively. A free resolution of $M \otimes N$ is given by $C_M \otimes_{\mathbb{C}} C_N/(Y = X')$. Tensoring this with R gives $C_M \otimes_{\mathbb{C}} C_N/(Y = X', X = Y'))$. $HH_*(M \otimes N)$ is the homology of this complex. Similarly, a free resolution of $N \otimes M$ is $C_M \otimes_{\mathbb{C}} C_N/(Y' = X)$, so $HH_*(M \otimes N)$ is also the homology of $C_M \otimes_{\mathbb{C}} C_N/(Y = X', X = Y'))$. □

The first property in the proposition says that HH_* behaves like a trace. In fact, Khovanov showed that HH_* categorifies the Jones-Ocneanu trace. If $\mathbb{B} \in \mathrm{SBim}_n$ we define

$$P(\mathbb{B}) = (aq)^{-n} \sum (-a^2q^2)^i \,\mathrm{qdim}\, HH^i(\mathbb{B}).$$

Theorem 5.7.5 ([43]). *If $\mathbb{B} \in \mathrm{SBim}_n$, $P(\mathbb{B}) = \mathrm{Tr}_n[\mathbb{B}]$, where $[\mathbb{B}]$ denotes the image of $\mathbb{B}$ in $K(\mathrm{SBim}_n) \simeq H_n$.*

Proof. It is enough to check the equality when $\mathbb{B}$ is a Bott-Samuelson bimodule. Since the Jones-Ocneanu trace satisfies and is determined by the MOY rules, it suffices to check that P satisfies the MOY rules as well. Rules 2) and 3) are a direct consequence of Propositions 5.6.3 and 5.6.5.

For MOY 0), suppose that $\mathbb{B}$ is a Bott-Samuelson bimodule in SBim_n, and let $\iota(\mathbb{B})$ be its image in SBim_{n+1}. If $C_{\mathbb{B}}$ is a free resolution of $\mathbb{B}$ over R_n^e, then

$$C_{\mathbb{B}} \otimes \left(q^2 R^e_{n+1} \xrightarrow{Y_{n+1}-X_{n+1}} R^e_{n+1} \right)$$

will be a free resolution of $\iota(\mathbb{B}_n)$ over Tr_{n+1}. After taking the tensor product with R_{n+1}, this becomes

$$(C_{\mathbb{B}} \otimes_{R_n^e} R_n) \otimes_{\mathbb{C}} \left(q^2 \mathbb{C}[X_{n+1}] \xrightarrow{0} \mathbb{C}[X_{n+1}] \right).$$

It follows that $HH^*(\iota(\mathbb{B})) = HH^*(\mathbb{B}) \otimes_{\mathbb{C}} \mathbb{C}[X_{n+1}] \otimes \Lambda^*(a_1)$ so, as desired,

$$P(\iota(\mathbb{B})) = a^{-1}q^{-1}P(\mathbb{B}) \left(\frac{q^2 - a^2q^2}{1-q^2} \right) = \{0\}P(\mathbb{B}).$$

The proof for MOY 1) is similar: $\iota(\mathbb{B}) \otimes \mathbb{B}_n$ has a free resolution

$$C_{\mathbb{B}} \otimes \left(q^2 R^e_{n+1} \xrightarrow{Y_n - Y' + Y_{n+1} - X_{n+1} pt} R^e_{n+1} \right) \otimes \left(q^4 R^e_{n+1} \xrightarrow{(X_{n+1}-Y_{n+1})(X_{n+1}-Y')} R^e_{n+1} \right).$$

After tensoring with $\mathbb{R}_n$, this becomes

$$C_{\mathbb{B}} \otimes_{R_n^e} \left(q^2 \mathbb{C}[Y'] \xrightarrow{Y_n - Y'} \mathbb{C}[Y'] \right) \otimes_{\mathbb{C}} \left(q^4 \mathbb{C}[X_{n+1}] \xrightarrow{0} \mathbb{C}[X_{n+1}]] \right).$$

As a chain complex over R_{n+1}, this is homotopy equivalent to the chain complex $C_{\mathbb{B}} \otimes (q^3 R_{n+1} \xrightarrow{0} q^{-1} R_{n+1})$, so

$$P(\iota(\mathbb{B}) \otimes \mathbb{B}_n) = (aq)^{-1} P(\mathbb{B}) \frac{q^4 - a^2q^2}{1-q^2} = \{1\}P(\mathbb{B}).$$ □

Proposition 5.7.6 ([67]). *If $\mathbb{B} \in \mathrm{SBim}_n$, then $HH_*(\mathbb{B})$ is free over R_n.*

Together with Theorem 5.7.5, the proposition says that $HH_*(\mathbb{B})$ can be easily computed using the MOY rules.

Example 5.7.7. In SBim_2, $\mathrm{Hom}(\mathbb{B}_1, \mathbb{B}_1) = HH_0(\mathbb{B}_1^2)$. Earlier, we computed that $\mathrm{Tr}_2(B_1^2) = [2]\{0\}\{1\}$, so

$$\mathrm{qdim}\,\mathrm{Hom}(\mathbb{B}_1, \mathbb{B}_1) = \frac{1+q^2}{(1-q^2)^2}.$$

Since $\mathrm{qdim}\, R_n = (1-q^2)^{-2}$, $\mathrm{Hom}(\mathbb{B}_1, \mathbb{B}_1)$ is free of rank 2 over R_2. The identity map and the map "multiplication by Y_1" are a basis for $\mathrm{Hom}(\mathbb{B}_1, \mathbb{B}_1)$ over R_2.

A similar computation shows that $\mathrm{Hom}(\mathbb{1}, \mathbb{B}_1) \simeq qR_2 \simeq \mathrm{Hom}(\mathbb{B}_1, \mathbb{1})$. We can describe the generators $S : \mathbb{1} \to \mathbb{B}_i$ and $S' : \mathbb{B}_i \to \mathbb{1}$ explicitly as follows. Identify $\mathbf{1} = R_2^e/\mathcal{I}_1$ and $\mathbb{B}_1 = R_2^e/\mathcal{I}_2$, where

$$\mathcal{I}_1 = (e_1, X_1 - Y_1) \quad \text{and} \quad \mathcal{I}_2 = (e_1, (Y_1 - X_1)(Y_2 - X_2))$$

with $e_1 = Y_1 + Y_2 - X_1 - X_2$. Clearly $\mathcal{I}_2 \subset \mathcal{I}_1$, and we define $S' : q\mathbb{B}_1 \to \mathbb{1}$ to be the quotient map. For the other direction, note that $(Y_1 - X_2)\mathcal{I}_1 \subset \mathcal{I}_2$, so there is a well-defined map $S : q\mathbb{1} \to \mathbb{B}_1$ given by $S(p) = (Y_1 - X_2)p$. Similarly, in SBim_n, we can define morphisms $S_i : q\mathbb{1} \to \mathbb{B}_i$ and $S_i' : q\mathbb{B}_i \to \mathbb{1}$ by acting on the variables with index i and $i+1$.

Exercise 5.7.8. Use the above method to compute $\mathrm{Hom}(\mathbb{B}_s, \mathbb{B}_{s'})$, where $s, s' \in S_3$. Describe generators for the Hom-spaces, and verify that Theorem 5.6.9 holds in this case.

5.8. The Rouquier complex We can now describe the HOMFLY-PT analog of the Khovanov complex of a tangle. Let $K^b(\mathrm{SBim}_n)$ denote the homotopy category of bounded complexes over SBim_n (not to be confused with the Grothendieck group $K(\mathrm{SBim}_n)$.)

Theorem 5.8.1 (Rouquier) [71]). *There is a well-defined map* $C : \mathrm{Br}_n \to K^b(\mathrm{SBim}_n)$ *satisfying*

$$C(\sigma_i) = \underline{q\mathbb{1}} \xrightarrow{S_i} \mathbb{B}_i \qquad C(\sigma_i^{-1}) = \mathbb{B}_i \xrightarrow{S_i'} \underline{q^{-1}\mathbb{1}}$$

and $C(\sigma\sigma') \sim C(\sigma) \otimes C(\sigma')$. *(The underlined terms in each complex are in homological grading* 0.*)*

Sketch of proof. The statement of the theorem clearly determines the complex assigned to a given braid word. To check that the theorem holds, we must show that braid words which correspond to the same braid have homotopy equivalent complexes. There are three things to check: $C(\sigma_i) \otimes C(\sigma_i^{-1}) \sim \mathbb{1}$, $C(\sigma_i\sigma_j) \sim C(\sigma_j\sigma_i)$ if $|i-j| > 1$, and $C(\sigma_1\sigma_2\sigma_1) \sim C(\sigma_2\sigma_1\sigma_2)$. The first relation corresponds to the second Reidemeister move, and its proof is entirely analogous to the proof of Proposition 4.9.3.

The second relation (far commutativity) is easy: if $|i-j| > 1$, then $B_iB_j = B_jB_i$, so $\mathbb{B}_i\mathbb{B}_j \sim \mathbb{B}_j\mathbb{B}_i$. The morphism spaces $\mathrm{Hom}(q\mathbb{B}_i, \mathbb{B}_i\mathbb{B}_j)$ and $\mathrm{Hom}(q\mathbb{B}_j, \mathbb{B}_i\mathbb{B}_j)$ are one-dimensional, so one easily checks that $C(\sigma_j\sigma_i) \simeq C(\sigma_i\sigma_j)$.

The proof of the braid relation is more complicated. Using Proposition 5.6.5, we see that $C(\sigma_1\sigma_2\sigma_1)$ is isomorphic to a complex of the form

$$q^3\mathbb{1} \to q^2(\mathbb{B}_1 \oplus \mathbb{B}_1 \oplus \mathbb{B}_2) \to q(\mathbb{B}_{12} \oplus \mathbb{B}_{21}) \oplus (1+q^2)\mathbb{B}_1 \to \mathbb{B}_w \oplus \mathbb{B}_1.$$

cancelling, we see this is homotopy equivalent to a complex

$$q^3\mathbb{1} \to q^2(\mathbb{B}_1 \oplus \mathbb{B}_2) \to q(\mathbb{B}_{12} \oplus \mathbb{B}_{21}) \to \mathbb{B}_w.$$

Using exercise 5.7.8, we see that all components of the differential belong to 1-dimensional morphism spaces. In fact, all possible components of the differential are nonzero. (This is best proved by using invariance under the second Reidemeister move to see that $C(\sigma_1\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}) \sim \mathbb{1}$.)

Finally, the relation $d^2 = 0$ can be used to show that any two complexes of the form above are isomorphic. Since the form of this complex is completely symmetric under the operation of permuting 1 and 2, it must be homotopy equivalent to $C(\sigma_2\sigma_1\sigma_2)$ as well. □

The Rouquier complex categorifies the homomorphism $\Psi : Br_n \to H_n$. If C is a chain complex over an additive category $\mathcal{C}$, we define $\chi(C) = \sum_i (-1)^i [C_i] \in K(\mathcal{C})$, where as usual $[C_i]$ denotes the image of the object $[C_i]$ in the Grothendieck group $K(\mathcal{C})$.

Proposition 5.8.2. *If* $\sigma \in Br_n$, $\chi(C(\sigma)) = \Psi(\sigma) \in H_n$.

Proof. Comparing with equation (5.4.1), we see that $\chi(C(\sigma_i^{\pm 1})) = \Psi(\sigma_i^{\pm 1})$. Since $\chi(C \otimes C') = \chi(C)\chi(C')$, the statement holds for all σ. □

Just as with the Khovanov complex of a tangle, positivity of the category SBim ensures that $C(\sigma)$ has a unique minimal representative which is well-defined up to isomorphism.

5.9. HOMFLY-PT homology The Hochschild homology HH_* is a covariant functor from $SBim_n$ to the category of modules over R_n, so $HH_*(C(\sigma))$ is a complex of R_n-modules. Its homology $H_*(HH_*(C(\sigma)))$ will have three gradings: the homological gradings on the complex and on Hochschild homology, and the q-grading. In keeping with our previous notation for the Hochschild homology and the Bar-Natan complex, we denote a shift in the homological grading by multiplication by t, and in the Hochschild grading by multiplication by a^2.

Definition 5.9.1. If $\sigma \in Br_n$, the HOMFLY-PT homology of its closure $\overline{\sigma}$ is defined to be $HHH(\overline{\sigma}) = t^{-n_+(\sigma)}(aq)^{-w(\sigma)}H_*(HH_*(C(\sigma)))$.

We index the homological gradings so that

$$HHH_{i,j}(\overline{\sigma}) = H_{i-n_+(\sigma)}(HH_{j-w(\sigma)}(C(\sigma))).$$

Theorem 5.9.2 (Khovanov-Rozansky [40, 44])**.** HHH *is a well-defined link invariant which categorifies the HOMFLY-PT polynomial, in the sense that*

$$\sum_{i,j}(-1)^i(-a^2q^2)^j \operatorname{qdim} HHH_{i,j}(L) = \overline{P}(L).$$

Sketch of proof. The relation with the Euler characteristic follows from Proposition 5.8.2, Theorem 5.7.5, and Definition 5.2.3. To prove that HHH is a link invariant, we must check that it is invariant under the Markov moves. Invariance under Markov 1) follows from Proposition 5.7.4. For invariance under Markov 2), we use the computation of $\mathrm{HH}(\iota(B))$ and $\mathrm{HH}(\iota(B)B_n)$ in the proof of Theorem 5.7.5 to see that $\mathrm{HH}(\sigma\sigma_n^{\pm 1}) \sim \mathrm{HH}(\sigma) \otimes (aq)^{-1}C_{\pm}$, where $C_{\pm}$ are the bicomplexes shown below, and $R = \mathbb{C}[X_{n+1}]$.

$$\begin{array}{ccc} qR & \xrightarrow{X_{n+1}} & q^{-1}R \\ {\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow \\ q^3R & \xrightarrow{1} & q^3R \end{array} \qquad\qquad \begin{array}{ccc} q^{-1}R & \xrightarrow{1} & q^{-1}R \\ {\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow \\ q^3R & \xrightarrow{X_{n+1}} & qR \end{array}$$

It is easy to see that $H_*(C_{\pm}) \simeq \mathbb{C}$. The homology is supported in the right-hand column of both bicomplexes, which is homological grading 1 for the positive crossing, but homological grading 0 for the negative one. This is accounted for by the shift by $t^{-n_+(\sigma)}$ in the definition of HHH. □

Alternate description: Expanding out the chain complex used to compute the Hochschild homology yields the following alternate description of the HOMFLY-PT homology. This is the original definition of Khovanov and Rozansky [40].

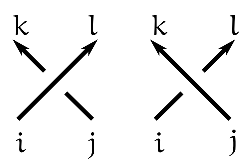


Figure 5.9.3. Positive and negative crossings

Suppose D is a planar braid diagram. Let $R = R(D) = \mathbb{C}[X_e]/(L_c)$, where e runs over the edges of D and (L_c) is an ideal generated by one linear relation L_c for each crossing c. If we let c_+ and c_- be positive and negative crossings with edges labeled as in the figure $L_{c_+} = L_{c_-} = X_i + X_j - X_k - X_l$. Since the L_c's are all linear polynomials, $R(D)$ is isomorphic to a polynomial ring.

We form a bicomplex

$$C(D) = \bigotimes_c C(c),$$

where the tensor product runs over all crossings of D. The bicomplexes $C(c)$ associated to the crossings c takes one of two forms, depending on whether c is positive or negative. Omitting the grading shifts, the diagram below shows the bicomplexes associated to c_+ (left) and c_- (right).

$$\begin{array}{ccc} R & \xrightarrow{(X_i-X_k)(X_i-X_l)} & R \\ {\scriptstyle 1}\uparrow & & {\scriptstyle X_i-X_l}\uparrow \\ R & \xrightarrow{X_i-X_k} & R \end{array} \qquad\qquad \begin{array}{ccc} R & \xrightarrow{X_i-X_k} & R \\ {\scriptstyle X_i-X_l}\uparrow & & {\scriptstyle X_i-X_l}\uparrow \\ R & \xrightarrow{(X_i-X_k)(X_i-X_l)} & R \end{array}$$

The maps in the horizontal direction are the Hochschild differentials, and the maps in the vertical direction correspond to differentials in the cube of resolutions. Up to an appropriate grading shift, we have

$$\mathrm{HHH}(\overline{\sigma}) = \mathrm{H}(\mathrm{H}(\mathrm{C}(\mathrm{D}), d_h), d_v).$$

In this formulation, we can form a reduced homology HHH^r by working over the ring $R_r(D) \subset R(D)$ generated by the differences $X_i - X_j$. If K is a knot, it can be shown that $\mathrm{HHH}^r(K)$ is finite dimensional, and that

$$\mathrm{HHH}(K) = \mathrm{HHH}^r(K) \otimes \mathrm{HHH}(\bigcirc).$$

5.10. Connections and Further Reading Khovanov's paper [44] is well worth reading. Elias and Williamson [21,22] developed a calculus for morphisms in the Soergel category which led to a proof of the Kazhdan-Lusztig conjectures in all types. More generally, the Kazhdan-Lusztig basis and the Soergel category paly a key role in the development of geometric representation theory and relate to many topics that are both beautiful and attractive to the geometrically inclined. The books by Humphreys [34] and Bjorner and Brenti [8] are good starting places.

In a different direction, we can consider the relation between HOMFLY-PT homology and Khovanov homology. The Jones polynomial is obtained from the HOMFLY-PT polynomial by substituting $a = q^2$. The categorified analog of such a substitution is a spectral sequence from HHH Kh [68]. In fact there is an entire family of such spectral sequences, corresponding to substituting $a = q^n$. The Alexander polynomial is obtained by substituting $a = q^0$, so it is natural to ask if there is a spectral sequence from HHH^r to $\widehat{HFK}$. Indeed Dowlin [18] recently proved that there is a spectral sequence from Kh_r to $\widehat{\mathrm{HFK}}$, which provides a (somewhat roundabout) way of constructing one.

The HOMFLY-PT homology of torus knots has many interesting connections to other areas of geometry and representation theory, including Hilbert schemes of plane curve singularities [58,60], Cherednik algebras [14,27,29], and the Hilbert scheme of $\mathbb{C}^2$, [28,59]. Their HHH was finally calculated through some remarkable work of Elias, Hogancamp and Mellit [20,32,53]

6. Λ^k colored polynomials

Witten [83] famously reformulated the Jones polynomial using quantum field theory. If Y is a 3-manifold equipped with a principal G-bundle E, the action of Witten's theory is given by the Chern-Simons functional on the space of connections on E. To get knot polynomials, we treat the knots as "Wilson loops" labeled by representations of G and take the expected value of a function based on the trace of the representation along the knot. To recover the Jones polynomial, we take $G = SU(2)$ and color every component with the vector (2-dimensional) representation. However Witten's formulation makes it clear that there are many other possibilities.

In mathematical terms, the theory that Witten created can be described as a relative 2+1 dimensional TQFT. (Indeed, Witten's work was a principle motivation for the definition of a TQFT.) Given a simple Lie algebra $\mathfrak{g}$, we form a category whose objects are closed surfaces Σ containing a set of points P which are *colored* (*i.e.* labeled) by representations of $\mathfrak{g}$. and whose morphisms are given by pairs $(Y, T) : (\Sigma_0, P_0) \to (\Sigma_1, P_1)$, where $Y : \Sigma_0 \to \Sigma_1$ is a cobordism and $T \subset Y$ is a colored tangle with ends on P_0 and P_1. We seek a functor from this category to the category of vector spaces and linear maps between them. So far, mathematicians are unable to give a rigorous definition of the path integral used in Witten's work, but Reshetikhin and Turaev [69] gave an algebraic construction of the invariants Witten described. They are now known as the WRT invariants.

Khovanov's program of categorification aims to upgrade these $2+1$ dimensional TQFT's to $2+1+1$ dimensional TQFT's. So far, most of its successes have been for polynomial invariants of knots and tangles in S^3 in type A (when $\mathfrak{g} = \mathfrak{sl}_n$.) We understand this process very well when all the colors are minuscule representations, *i.e.* exterior powers of the vector representation. In this situation, there are many different ways to categorify, and they are more or less all known to agree. For other colors, there are several different possible categorifications, not all of which are the same. Outside of type A, very little is known, and essentially nothing is known about categorifications for knot and tangles in 3-manifolds other than S^3. The exception comes from the world of Floer homology, where knot Floer homology, which categorifies the Alexander polynomial, fits into the 3+1 dimensional TQFT provided by Heegaard Floer homology. (For more information on this topic, see Hom's lectures in this volume.)

6.1. The yoga of WRT In order to define the WRT invariants, we must specify another parameter, known as the *level*. In general, the dependence of the invariants on the level is complicated, but in the case of tangles in B^3, it is quite simple. Instead of working over $\mathbb{C}$, we work over the ring $\mathbb{Z}[q^{\pm 1}]$. To recover the invariants at a particular level, we substitute $q = \omega$, where ω is some root of unity determined by the level.

We outline the basic properties of the WRT invariants in this setting. Fix a simple Lie algebra $\mathfrak{g}$. We consider the category $\mathfrak{g}\mathbf{Tan}$ of $\mathfrak{g}$-colored, oriented tangles in $\mathbb{R}^2 \times [0, 1]$. The objects of this category are triples of the form $\mathbf{X} = (X_n, \mathbf{s}, \mathbf{V})$, where X_n is our preferred set of n points in $\mathbb{R}^2$, $\mathbf{s} \in \{\pm\}^n$ is an n-tuple of orientations on these points, and $\mathbf{V} \in (\mathrm{Rep}(\mathfrak{g}))^n$ is an n-tuple of representations labeling the points of X_n. We divide out by the equivalence relation which allows us to reverse the orientation s_i on the point x_i at the cost of replacing its label V_i with the dual representation V_i^*.

The morphisms in $\mathfrak{g}\mathbf{Tan}$ are oriented, colored tangles $T \subset \mathbb{R}^2 \times [0, 1]$. If T is such a tangle, then $\partial T = \partial_0 T \amalg \partial_1 T$, where $\partial_i T \subset \mathbb{R}^2 \times i$. The points of ∂T inherit the orientations and colors of the tangle components which they lie on. In this way, we can view T as an element of $\mathrm{Mor}(\partial_0 T, \partial_1 T)$. Again, we divide out be the relation which allows us to replace the orientation on a component of T at the cost of replacing the color with the dual color. As usual, we also divide out by the action of isotopies on $\mathbb{R}^2 \times I$ which fix $\mathbb{R}^2 \times \partial I$.

To a pair of objects $\mathbf{X}$ and $\mathbf{X}'$ of $\mathfrak{g}\mathbf{Tan}$, the WRT invariant associates a free $\mathbb{Z}[q^{\pm 1}]$ module $\mathbf{W}_{\mathbf{X},\mathbf{X}'}$. To a tangle $T : \mathbf{X} \to \mathbf{X}'$ it should associate an element $\langle T\rangle_{\mathfrak{g}} \in \mathbf{W}_{\partial_0 T, \partial_1 T}$, which is well-defined up to multiplication by $\pm q^k$. (As with the Kauffman bracket, this ambiguity can be fixed by considering framed tangles.) If $T : \mathbf{X}' \to \mathbf{X}''$ is another tangle, there should be a composition map

$$\cdot : \mathbf{W}_{\mathbf{X},\mathbf{X}'} \times \mathbf{W}_{\mathbf{X}',\mathbf{X}''} \to \mathbf{W}_{\mathbf{X},\mathbf{X}''}$$

which satisfies

$$\langle TT'\rangle_{\mathfrak{g}} \sim \langle T\rangle_{\mathfrak{g}} \cdot \langle T'\rangle_{\mathfrak{g}}$$

If $\mathbf{X}_\emptyset := (X_0, \emptyset, \emptyset)$, then $\mathbf{W}_{\mathbf{X}_\emptyset,\mathbf{X}_\emptyset} = \mathbb{Z}[q^{\pm 1}]$. If $L \subset \mathbb{R}^3$ is a closed, oriented, colored link, then we can view L as a tangle $L : \mathbf{X}_\emptyset \to \mathbf{X}_\emptyset$, so $\langle L\rangle_{\mathfrak{g}} \in \mathbb{Z}[q^{\pm 1}]$. We have already seen examples of these invariants. The polynomial $\overline{P}_N(L) := \overline{P}(L)|_{a=q^N}$ is the WRT invariant corresponding to $\mathfrak{g} = \mathfrak{sl}_N$ in which all components of L are colored with the vector representation of $\mathfrak{sl}_N$.

Example 6.1.1. Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathbf{X}_n$ is X_n, with all points labeled with the vector representation V. The choice of orientation is irrelevant, since $V \simeq V^*$. In this case, $\mathbf{W}_{\mathbf{X}_n,\mathbf{X}_m} = V_{n,m}$ is the linear space spanned by crossingless planar tangles (see section 4.3).

If T is a tangle whose components are labeled with the vector representation, then $\langle T\rangle_{\mathfrak{sl}_2}$ is the Kauffman bracket of T.

The dimension of $\mathbf{W}_{\mathbf{X},\mathbf{X}'}$ can be determined from the representation theory of $\mathfrak{g}$. To the object $\mathbf{X} = (X_n, \mathbf{s}, \mathbf{V})$, we assign

$$R(\mathbf{X}) = \bigotimes_{i=1}^{n} V_i^{s_i} \in \mathrm{Rep}(\mathfrak{g}),$$

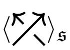
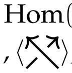
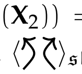
where $V_i^+ = V$ and $V_i^- = V^*$. Then

$$\dim_{\mathbb{Z}[q^{\pm 1}]} \mathbf{W}_{\mathbf{X},\mathbf{X}'} = \dim_{\mathbb{C}} \mathrm{Hom}(R(\mathbf{X}), R(\mathbf{X}')).$$

If V and V' are irreps of $\mathfrak{g}$, then

$$\dim(V, V') = \begin{cases} 1 & \text{if } V \simeq V' \\ 0 & \text{if } V \not\simeq V' \end{cases}$$

Hence this dimension can be computed by decomposing $R(\mathbf{X})$ and $R(\mathbf{X}')$ into their irreducible components.

Example 6.1.2. Let $\mathfrak{g} = \mathfrak{sl}_N$, and let $\mathbf{X}_2$ consist of two positively oriented points, each labeled with V. Then $R(\mathbf{X}_2) = V \otimes V \simeq \mathrm{Sym}^2 V \oplus \Lambda^2 V$ splits as a sum of two different irreps, so $\dim \mathrm{Hom}(R(\mathbf{X}_2), R(\mathbf{X}_2)) = 2$. Hence there must be a linear relation between $\langle$$\rangle_{\mathfrak{sl}_N}$, $\langle$$\rangle_{\mathfrak{sl}_N}$, and $\langle$$\rangle_{\mathfrak{sl}_N}$. This is the source of the HOMFLY-PT skein relation.

Exercise 6.1.3. Let $\mathfrak{g} = \mathfrak{sl}_2$, and let V be the vector representation. Show that

$$\dim \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) = \begin{cases} C_{n+m/2} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd} \end{cases},$$

where

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

is the kth Catalan number. Compare with Exercise 4.2.2.

Exercise 6.1.4. Let $\mathfrak{g} = \mathfrak{sl}_N$, where $N \geqslant 4$. If you know some representation theory, show that $\Lambda^2 V \otimes \Lambda^2 V$ has three irreducible summands. More generally, show that $\Lambda^k V \otimes \Lambda^l V$ has $\min(k+1, l+1)$ irreducible summands when $N \geqslant k+l$?

6.2. Webs We now discuss the *MOY state model*, which was introduced by Murakami, Ohtsuki, and Yamada in [55]. It gives a simple way of constructing the invariants of the previous section when $\mathfrak{g} = \mathfrak{sl}_N$ and all colors are exterior powers of the vector representation. It should be viewed as a generalization of the construction of the Kauffman bracket in section 4.

We start by defining a diagrammatic category $\mathcal{W}$ which will play the same role that the category of planar tangles played in the definition of the Kauffman bracket. Let $X_n = \{x_1, \ldots, x_n\}$ be our standard set of n points in $\mathbb{R}$.

Definition 6.2.1. An (n, n')–*web* is an oriented graph embedded in $\mathbb{R} \times I$ such that

(1) $\Gamma \cap (\mathbb{R} \times 0) = X_n$, $\Gamma \cap (\mathbb{R} \times 1) = X_{n'}$.
(2) Each edge e of Γ is labeled by a non-negative integer $k(e)$.
(3) $X_n \cup X_{n'}$ is the set of univalent vertices of Γ.
(4) The other vertices of Γ are locally modeled on one of the two pictures shown in the figure below.

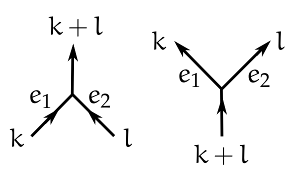


Figure 6.2.2. Internal vertices of a web

Example 6.2.3. The MOY graphs considered in the previous lecture are webs, where $k(e) = 1$ if e is a thin edge, and 2 if e is a thick edge.

If W is an (n, n')–web, its restriction to $\mathbb{R} \times 0$ determines a triple $\mathbf{X} = (X_n, \mathbf{s}, \mathbf{k})$ where s_i is the induced orientation at x_i, and $k_i = k(e_i)$, where e_i is the unique edge adjacent to x_i. The objects of $\mathcal{W}$ are such triples, and the morphisms are webs between them. Composition is given by horizontal stacking, just as in the category of planar tangles.

6.3. The MOY bracket A *closed web* W is a web with no univalent vertices; it can be viewed as a morphism $W : \mathbf{X}_\emptyset \to \mathbf{X}_\emptyset$. The key construction of [55] is a state model for evaluating the $\mathfrak{sl}_N$ bracket of a closed web.

Definition 6.3.1. Let Γ be a closed web. A $\mathfrak{sl}_N$*–state* of Γ is a function which assigns to each edge e of Γ a subset $A_e \subset \{-N+1, -N+3, \dots, N-1\}$ such that

(1) $|A_e| = k(e)$ is the label on e.
(2) At each vertex v of Γ,
$$\bigcup_{v\to e} A_e = \bigcup_{e \to v} A_e.$$

One side of the equation in condition (2) is a union of two sets, while the other side is a single set. From the balancing condition on the labels at trivalent vertices, we see that the two sets appearing in the union must be disjoint.

The definition of a state has a simple physical interpretation. We think of Γ as being a wiring diagram for a collection of electric cables, where the number of cables passing through e is exactly $k(e)$. Each cable is labeled by an element of $\{-N+1, -N+3, \dots, N-1\}$; property (2) tells us that no cables are created or destroyed as we pass through a vertex.

A state σ of Γ determines a resolution Γ_σ, which is a collection of closed oriented circles in $\mathbb{R}^2$. Each circle C is labeled by $\sigma_C \in \{-N+1, -N+3, \dots, N-1\}$. In the physical model above, the circles are the closed loops formed by individual cables.

We now define two weights associated to a state σ. The first is
$$R(\sigma) = \sum_{C \in \Gamma_\sigma} \sigma_C \operatorname{rot} C$$
where rot C is the rotation number: $+1$ if C is counterclockwise, and -1 if C is clockwise.

To define the second quantity, suppose that v is a vertex of Γ. After rotating the plane of the paper, v has one of the two forms shown in Figure 6.2.2. We define
$$w(\sigma, v) = \sum_{a_1 \in A_{e_1}, a_2 \in A_{e_2}} \frac{1}{2} s(a_1, a_2),$$
where
$$s(a,b) = \begin{cases} 1 & a > b \\ -1 & a < b \end{cases}.$$
The second weight is $W(\sigma) = \sum_{v \in \Gamma_\sigma} w(\sigma, v)$.

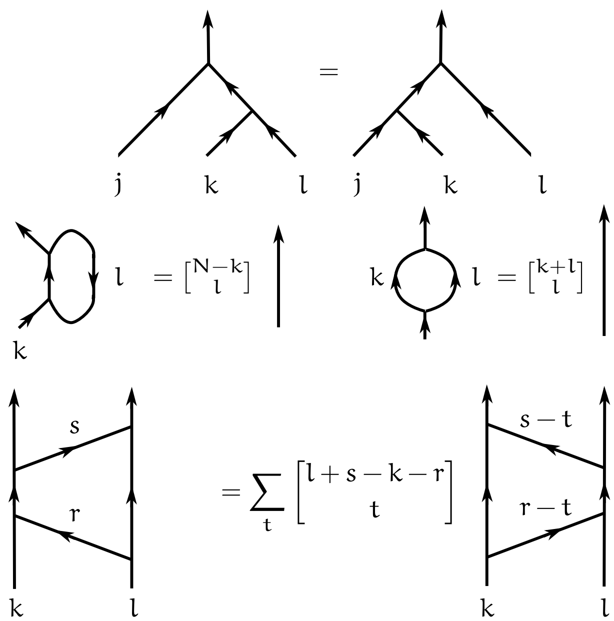
Definition 6.3.2. If Γ is a closed web, we define the $\mathfrak{sl}_N$ bracket
$$\langle \Gamma \rangle_N = \sum_\sigma q^{W(\sigma)+R(\sigma)}$$
where the sum runs over all $\mathfrak{sl}_N$–states of Γ.

Exercise 6.3.3. Show that if Γ is a single circle labeled by k, $\langle \Gamma \rangle_N = \begin{bmatrix} N \\ k \end{bmatrix}$, where $\begin{bmatrix} N \\ k \end{bmatrix}$ is the *quantum binomial coefficient*:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$$

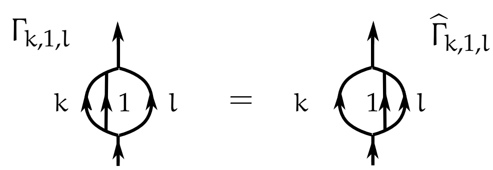
and $[n]! = [1][2]\cdots[n]$.

Proposition 6.3.4 ([55]). *The MOY bracket satisfies the following relations:*



As an example, we prove the right-hand relation in the middle row, which is a generalization of the the MOY 2) move. Let $\Gamma_{k,l}$ and Γ be the graphs on the left and right-hand sides of the equation. We prove the statement by induction on l. For the base case, take $l = 1$. Let e be the edge of Γ shown in the figure and let e_1, e_2 bet the edges of $\Gamma_{k,l}$ labeled k and l. A state σ' of Γ' is specified by a state σ of Γ together with an element $a_{e_2} \in A_e$. We have $W(\sigma') = W(\sigma) + n - (k - n)$, where n is the number of elements of the set $\{a \in A_e \,|\, a < a_{e_2}\}$. Since $R(\sigma') = R(\sigma)$, it follows that $\langle \Gamma_{k,1} \rangle_N = [k]\langle \Gamma \rangle_N$.

For the general case, consider the two webs $\Gamma_{k,1,l}$ and $\widehat{\Gamma}_{k,1,l}$ shown below.



Using the MOY relation in the top row above, we see that $\langle \Gamma_{k,1,l} \rangle_N = \langle \widehat{\Gamma}_{k,1,l} \rangle_N$. Next, we use the relation for $l = 1$ to see that

$$[k+1]\langle \Gamma_{k+1,l} \rangle_N = \langle \Gamma_{k,1,l} \rangle_N = \langle \widehat{\Gamma}_{k,1,l} \rangle_N = [l+1]\langle \Gamma_{k,l+1} \rangle_N.$$

By the induction hypothesis, we see that

$$\langle \Gamma_{k,l+1}\rangle_N = \frac{[k+1]}{[l+1]}\langle \Gamma_{k+1,l}\rangle_N = \frac{[k+1]}{[l+1]}\begin{bmatrix} k+l+1 \\ l \end{bmatrix}\langle \Gamma\rangle_N = \begin{bmatrix} k+l+1 \\ l+1 \end{bmatrix}\langle \Gamma\rangle_N.$$

The relations of Proposition 6.3.4 are generalizations of the MOY relations used in section 5. Note that the coefficients that appear can all be expressed as rational functions of $a := q^N$ and q. For example

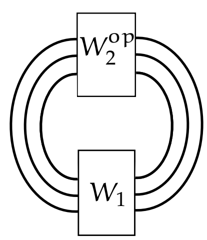
$$\begin{bmatrix} N \\ k \end{bmatrix} = \frac{\{0\}\{1\}\cdots\{k-1\}}{[1][2]\ldots[k]}.$$

This can be used to prove that as N varies, the polynomials $\langle \Gamma\rangle_N$ are all specializations of a rational function of a and q.

Proposition 6.3.5 ([84]). *If Γ is a closed web, there is a rational function $\langle \Gamma\rangle \in \mathbb{Z}[a^{\pm 1}](q)$ such that $\langle \Gamma\rangle_N = \langle \Gamma\rangle|_{a=q^N}$ for all $N > 0$.*

6.4. The web category For motivation, we return to the Kauffman bracket. Associated to the objects X_n, X_m, we had a vector space $V_{n,m}$ generated by simple planar tangles. As in exercise 4.4.3, we can view $V_{n,m} = \mathrm{Hom}_{\mathbf{P}}(X_n, X_m)$, where **P** is the quotient of the category of planar tangles up to isotopy by the ideal generated by the local relation $\bigcirc = (q+q^{-1})$.

We would like to do something similar with the category $\mathcal{W}$ to obtain a category **Web** for which $\mathrm{Hom}(\mathbf{X}, \mathbf{X}')$ is a finitely generated R-module. One approach might be to try to take the quotient of $\mathrm{Add}(\mathcal{W}) \otimes R$ by the ideal generated by the MOY relations. A more elegant method is to use the following universal construction, which is due to Blanchet, Habegger, Masbaum, and Vogel [9].



Let $\widetilde{\mathbf{W}}_{\mathbf{X},\mathbf{X}'} := \mathrm{Hom}_{\mathrm{Add}(\mathcal{W})\otimes R}(\mathbf{X}, \mathbf{X}')$ be the free R-module generated by all webs $W : \mathbf{X} \to \mathbf{X}'$. We define a pairing $(\cdot,\cdot) : \widetilde{\mathbf{W}}_{\mathbf{X},\mathbf{X}'} \times \widetilde{\mathbf{W}}_{\mathbf{X},\mathbf{X}'} \to R$ by

$$(W_1, W_2) = \langle \overline{W_1 W_2^{op}} \rangle$$

where $\overline{W_1 W_2^{op}}$ is the closed web shown in the figure above, and extending linearly to $\widetilde{\mathbf{W}}$. Let

$$I_{\mathbf{X},\mathbf{X}'} = \{W \in \widetilde{\mathbf{W}}_{\mathbf{X},\mathbf{X}'} \mid (W, W') = 0 \text{ for all } W' \in \widetilde{\mathbf{W}}_{\mathbf{X},\mathbf{X}'}\}.$$

Lemma 6.4.1. *The $I_{\mathbf{X},\mathbf{X}'}$ form an ideal $\mathcal{I}$ in $\mathrm{Add}(\mathcal{W}) \otimes R$.*

Proof. From the figure, we see that $(W_1W_2, W_3) = (W_1, W_3W_2^r)$. If $W_1 \in I_{\mathbf{X},\mathbf{X}'}$, and $W_2 : \mathbf{X}' \to \mathbf{X}''$, then $(W_1W_2, W_3) = (W_1, W_3W_2^r) = 0$, so $W_1W_2 \in I_{\mathbf{X},\mathbf{X}'}$. A similar argument shows that $\mathcal{I}$ is also a right ideal. □

Definition 6.4.2. **Web** $= \mathcal{W} \otimes R/\mathcal{I}$.

Let $\mathbf{X}(k,l)$ be the object consisting of two positively oriented points with colors k and l respectively, and let $\mathbf{W}(k,l) = \mathrm{Hom}_{\mathbf{Web}}(\mathbf{X}(k,l), \mathbf{X}(k,l))$.

Proposition 6.4.3. $\dim \mathbf{W}(k,l) = \min(k+1, l+1)$.

6.5. The Λ^k colored HOMFLY-PT polynomial We defined the Kauffman bracket for tangles by assigning to a tangle diagram $D : X_n \to X_m$ its bracket $\langle D \rangle \in V_{n,m}$. Similarly, a colored oriented tangle diagram $D : \mathbf{X} \to \mathbf{X}'$ should determine an element $\langle D \rangle \in \mathbf{W}_{\mathbf{X},\mathbf{X}'}$.

As we did with the Jones polynomial, we start with the diagram of a single crossing. Let $\nwarrow\!\!\!\!\nearrow_{k,l}$ be the diagram of a positive crossing, where the overstrand is labeled by k, and the understrand is labeled by l. Similarly, let $\nwarrow\!\!\!\!\nearrow'_{k,l}$ be the diagram of a negative crossing where the understrand is labeled by k and the overstrand is labeled by l.

We should have $\langle \nwarrow\!\!\!\!\nearrow_{k,l} \rangle \in \mathbf{W}(k,l)$. By Proposition 6.4.3, we know this space has dimension $\min(k+1, l+1)$. Our first task is to write down a nice basis for this space.

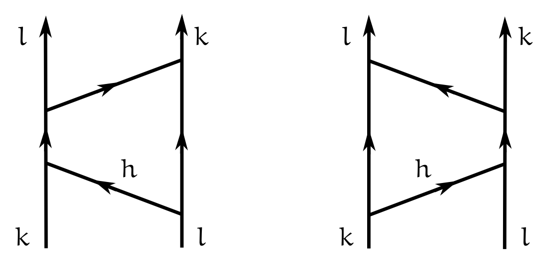


Figure 6.5.1. Basis webs for $\mathbf{W}(k,l)$

If $k \geqslant l \geqslant h \geqslant 0$, let $W_{k,l;h}$ be the open web shown in the figure to the left; while if $l \geqslant k \geqslant h \geqslant 0$, let $W'_{k,l;h}$ be the open web shown in the figure to the right.

Definition 6.5.2. If $k \geqslant l$, we define

$$\langle \nwarrow\!\!\!\!\nearrow_{k,l} \rangle = \sum_{h=0}^{l} (-q)^{h-l} \langle W_{k,l;h} \rangle \qquad \langle \nwarrow\!\!\!\!\nearrow'_{k,l} \rangle = \sum_{h=0}^{l} (-q)^{l-h} \langle W_{k,l;h} \rangle$$

Similarly, if $k \leqslant l$, we define

$$\langle \nwarrow\!\!\!\!\nearrow_{k,l} \rangle = \sum_{h=0}^{k} (-q)^{h-k} \langle W'_{k,l;h} \rangle \qquad \langle \nwarrow\!\!\!\!\nearrow'_{k,l} \rangle = \sum_{h=0}^{k} (-q)^{k-h} \langle W'_{k,l;h} \rangle$$

In the case where $k = l = 1$, these relations reduce to the previously known ones for the HOMFLY-PT polynomial:

$$\langle \nwarrow\!\!\!\!\nearrow \rangle = q \langle)(\rangle - \langle \Upsilon \rangle \quad \text{and} \quad \langle \nwarrow\!\!\!\!\nearrow' \rangle = q^{-1} \langle)(\rangle - \langle \Upsilon \rangle.$$

Exercise 6.5.3. If $k = l$, use MOY relations to show that the two definitions are equal.

If $D : \mathbf{X} \to \mathbf{X}'$ is a colored oriented tangle diagram Definition 6.5.2 expresses $\langle D \rangle$ as a sum of webs. The MOY bracket $\langle D \rangle$ is the image of this sum in $\mathbf{W}_{\mathbf{X},\mathbf{X}'}$. Just like the Kauffman bracket, the MOY bracket is invariant under Reidmeister moves 2) and 3), but not Reidemeister 1).

If D is a colored link diagram, we define $\mathbf{w}(D) = \sum_i k_i w(L_i)$, where the sum runs over the components L_i of D.

Theorem 6.5.4 ([55])**.** *If* D *and* D' *are related by a Reidemeister move, then we have* $a^{-k\mathbf{w}(D)}\langle D \rangle = a^{-k\mathbf{w}(D')}\langle D' \rangle$.

Exercise 6.5.5. Check that the theorem holds for the Reidemeister 1) move.

Definition 6.5.6. If L is an integer-colored link, $\overline{P}(L) := a^{-k\mathbf{w}(D)}\langle D \rangle \in \mathbb{Z}[a^{\pm 1}](q)$ is the unnormalized Λ-colored HOMFLY-PT polynomial of L.

When all $k_i = 1$ $\overline{P}(L)$ is the HOMFLY-PT polynomial as defined in the previous section. $\overline{P}(L)|_{a=q^N}$ is the WRT invariant of L associated to the Lie algebra $\mathfrak{sl}_N$, where each component of L is colored by the representation $\Lambda^{k_i}V$. If K is a knot with color k, we write $\overline{P}(K) = \overline{P}^{\Lambda_k}(K)$. In this case, we can define the normalized polynomial

$$P^{\Lambda^k}(K) = \frac{\overline{P}^{\Lambda^k}(K)}{\overline{P}^{\Lambda^k}(\bigcirc)},$$

and $P^{\Lambda^k}(K) \in \mathbb{Z}[q^{\pm 1}, q^{\pm 1}]$.

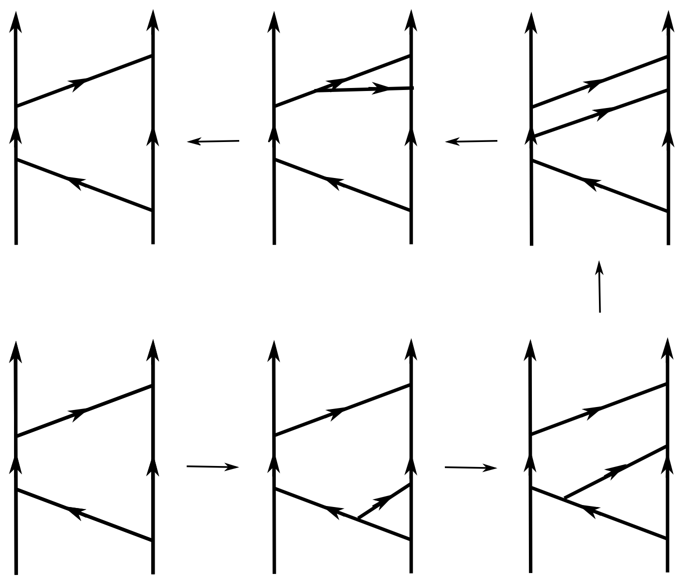
6.6. Categorification Having defined the Λ^k-colored $\mathfrak{sl}_N$ polynomials, we briefly discuss the problem of categorifying them. Again, we seek to imitate the constructions of section 4. The first step is to find the analog of the 2-category **Cob**.

Definition 6.6.1. A *pre-foam* is a 2-dimensional finite cell complex F embedded in $\mathbb{R} \times I \times I$ together with a labeling of every two-dimensional face f by an orientation, a non-negative integer $k(f)$, and a polynomial $p(f)$, which is a symmetric polynomial in $k(f)$ variables. F should satisfy the following conditions:

(1) $F \cap \mathbb{R} \times I \times 0 = W_0 : \mathcal{X} \to \mathbf{X}'$ and $F \cap \mathbb{R} \times I \times 1 = W_1 : \mathcal{X} \to \mathbf{X}'$ are webs.
(2) $F \cap \mathbb{R} \times 0 \times I = \mathbf{X} \times I$ and $F \cap \mathbb{R} \times 1 \times I = \mathbf{X}' \times I$
(3) In the interior of $\mathbb{R} \times I \times I$, F is locally modeled on the following pictures:
   (a) A plane labeled by $j \in \mathbb{N}$.
   (b) $Y \times I$, where Y is a trivalent vertex with edges labeled by j, k and $j + k$.
   (c) The cone on the 1-skeleton of a tetrahedron, with the six edges labeled by $j, k, l, j + k, j + l, j + k + l$.

The polynomial $p(f)$ fills the role of the "dots" in **Cob**. There is a 2-category **PFoam** of pre-foams, in which the objects are colored oriented points **X**, in which the 1-morphisms are webs $W : \mathbf{X} \to \mathbf{X}'$, and in which the 2-morphisms are pre-foams $F : W_0 \to W_1$. Compositions are given by horizontal and vertical stacking.

In **PFoam**, there are foams $F_+ : W_{k,l,h} \to W_{k,l,h+1}$ and $F_- : W_{k,l,h} \to W_{k,l,h-1}$. The foam F_+ is illustrated by the movie in the figure below. To get the foam F_-, we run the movie in reverse. Reflecting across a vertical line gives foams $F'_+ : W'_{k,l,h} \to W'_{k,l,h+1}$ and $F'_- : W'_{k,l,h} \to W'_{k,l,h-1}$.



Definition 6.6.2. If $g \geqslant l$, the *Rickard complex* associated to the crossing $\nwarrow\!\!\nearrow_{k,l}$ $(k \geqslant l)$ is

$$\langle \nwarrow\!\!\nearrow_{k,l} \rangle = q^l W_{k,l;0} \xrightarrow{F_+} q^{(l-1)} W_{k,l;1} \xrightarrow{F_+} \dots \xrightarrow{F_+} W_{k,l;l}.$$

There are similar formulas for the other configurations of crossings.

Note that the Rickard complex is not actually a complex in **PFoam**; there is no reason that $d^2 = 0$. To make it into one, we must pass to a new category **Foam**, which is a quotient of **PFoam** — just as we passed from **Cob** to Bar-Natan's category. The resulting category should also have enough relations to ensure that the MOY relations categorify to isomorphisms, in the same way that we were able to prove $\bigcirc \simeq (1 + q^{-1})$ in $\mathcal{C}^{BN}$. A careful list of local relations which must hold in **Foam** may be found in [65]. There are many ways to construct such a category: using matrix factorizations [45], [84], representation theory [51], [78], derived categories of sheaves [13], or categorified quantum groups [65]. Perhaps the simplest construction is due to Robert and Wagner [70]. They define a state sum formula for evaluating a closed foam, analogous to the MOY state sum formula for evaluating a closed web, and show that it satisfies the relations in [65]. The quotient category **Foam** can then be constructed using the BHMV method [9], just as we constructed **Web** from **PWeb**.

6.7. Connections and Further Reading Just as the reduced $\mathfrak{sl}_N$ homologies of a knot limit to its reduced HOMFLY-PT homology as $N \to \infty$, the reduced Λ^k colored homologies limit to a well-defined Λ^k-colored HOMFLY-PT homology $H^{\Lambda^k}(K)$. There are many conjectures in the physics literature about the structure of these invariants. We mention two of the most interesting. The first is the exponential growth conjecture of Gukov and Stosic [31], which states that $\dim H^{\Lambda^k}(K) \geqslant \dim(HHH^r(K))^k$. This was proved by Wedrich [81], who studied deformations and differentials on the Λ^k colored homology. The second is the knots-quivers correspondence [49]. This remarkable conjecture states that all the polynomials $P^{\Lambda^k}(K)$ can be recovered from a finite amount of data, in the form of a vector space equipped with a quadratic form and two linear forms. The conjecture is known to hold for all arborescent knots [76].

In a different direction, we could consider categorifying the WRT invariants for arbitrary colors in type A. A standard method for defining the WRT invariants is via Jones-Wenzl projectors. A Jones-Wenzl projector is a central idempotent in the Hecke algebra H_n; the set of such idempotents is naturally in bijection with the set of partitions of n. As is well-known, any such partition λ determines a representation V_λ of $\mathfrak{sl}_N$. If L is a link whose components L_i are labeled by partitions λ_i, we form a new link $\widehat{L}$ by taking the $|\lambda_i|$-strand cable of each L_i and "inserting the projector" e_{λ_i} into it. (More precisely, we take a braid representative of $\widehat{L}$, look at its image in the Hecke algebra, and insert a factor of e_{λ_i} into each clutch of $|\lambda_i|$ strands.) The trace of the resulting Hecke algebra element is the $\vec{\lambda}$ colored HOMFLY-PT polynomial of L.

A beautiful theorem of Rozansky [72] says that if we take an approriate limit of $CKh(T(n, m))$ as $m \to \infty$, the result is a categorification of the Jones-Wenzl projector in TL_n which corresponds to the representation $\mathrm{Sym}^n(V)$. Similar categorified projectors exist in the Soergel category and were studied by Hogancamp [33] and Abel-Hogancamp [1], who computed the stable HOMFLY-PT homology of the torus knots, as first conjectured in [19]. They can be used to define colored versions of the HOMFLY-PT homology but be careful: using this approach to define a Λ^k colored homology group results in an invariant which is different from those described above.

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