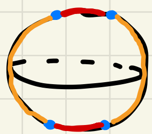
Rational Knots + Rational Tangles

Rational Tangles:

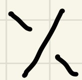
$P \subset S^2$ 4 points

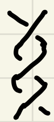
$T : (I \amalg I, \partial) \hookrightarrow (D^3, P)$
is a 2-strand tangle

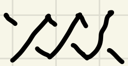
$Mod(S^2, P) \circlearrowright \{\text{2-strand tangles}\}$
$\cup$
$G_0 = \langle \tau_a, \tau_b \rangle$

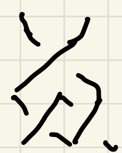


$\boxed{\begin{array}{l} \tau_a \cdot T_{p/q} = T_{\frac{p+q}{q}} \\ \tau_b^{-1} \cdot T_{p/q} = T_{\frac{p}{p+q}} \end{array}}$

$T_{1/1} =$ 

$T_{3/1} =$ 

$T_{1/3} =$ 

$T_{3/2} = \tau_a \cdot T_{1/2} =$ 

Ex:  $= T_{0/1}$

Shubert
Conway

$\{\text{rational tangles}\} = G \cdot T_{0/1} = G_0 \cdot T_{0/1}$

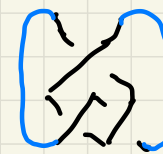
Classified by $p/q \in \mathbb{Q} \cup \{\infty\}$

Branched double cover $\Sigma(D^3, T_{p/q}) = S^1 \times D^2$

$m = \Sigma(a) \quad \ell = \Sigma(b) \qquad [\partial D^2] = pm + q\ell$

Rational Knot $K_{p/q} = T_{p/q} \cup T_{0/1}$

p odd



$K_{5/2}$

$\Sigma(K_{p/q}) = L(p,q)$

$\Rightarrow \det K_{p/q} = p$

Thm: $\overline{H}(K_{p/q})$ is thin: there's a linear relation between a, q and homological gradings.

$\Rightarrow$ $\overline{H}(K_{p/q})$ is determined by $\widetilde{P}(K_{p/q})$ $\quad \sigma(K_{p/q})$

$P(K_{p/q})$ is alternating: $P(K_{p/q}) = \pm \sum n_{ij} a^{2i}(-q)^{2j}$ $\qquad n_{ij} \geq 0$

$\dim \overline{H}(K_{p/q}) = \det K_{p/q} = p$

Ex: $$P(K_{5/2}) = \begin{array}{ccc} & +a^{2} & \\ -q^{-2} & +1 & -q^{2} \\ & +a^{-2} & \end{array} \qquad \Big\downarrow a$$

$$\xrightarrow{\qquad} q$$

NB: K alternating $\Rightarrow$ $\widehat{HFK}(K)$, $Kh^{\circ}(K)$ thin

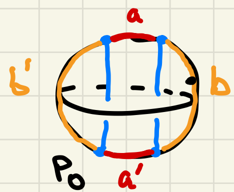
$\not\Rightarrow$ $\overline{H}(K)$ thin

$\exists$ K alternating w/ $P(K)$ not alternating

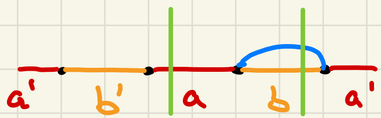
$P_0 \in P$ is point fixed by G_0

unmasked component of $T_{p/q}$

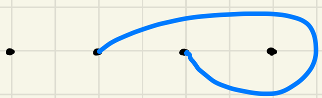
is isotopic to $L_{p/q} \subset S^2$

Ex: 

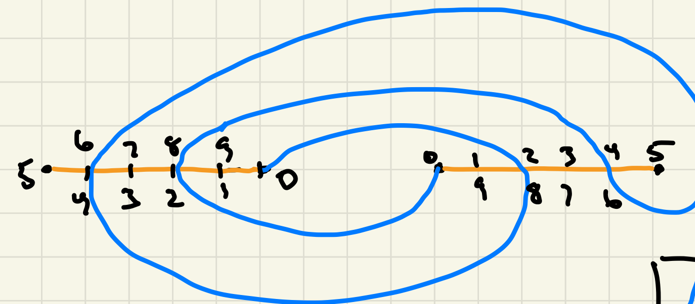
$L_{0/1} = b$

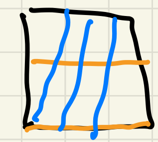
$L_{0/1}$: 

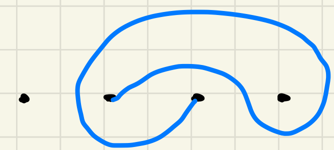
$L_{1/1}$: 

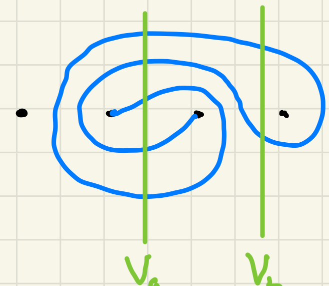
$L_{1/2}$: 

Easier way to draw:





$L_{3/2}$: 

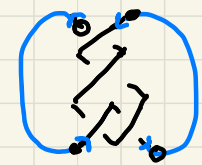
$L_{5/2}$: 

$|V_a \cap L_{p/q}| = p$

$|V_b \cap L_{p/q}| = q$

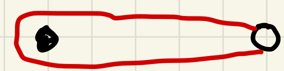
Philosophy: $L_{p/q}$ determines invariants of $T_{p/q}$, $K_{p/q}$

Ex: Orient $K_{p/q}$ inwards at P_0.



Weights at P_i determined by orientation

Inwards : $a^2 q^{-2}$ ●
Outwards : q^2 ○



$M_{0/1}$ = double of $L_{0/1}$ open at outward end

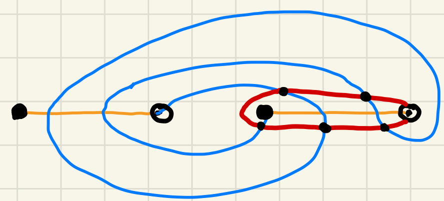
$S = L_{p/q} \cap M_{0/1}$

Prop: $\tilde{P}(K_{p/q}) \sim \sum_{x \in S} \operatorname{sign} x \; a^{a(x)} q^{q(x)}$

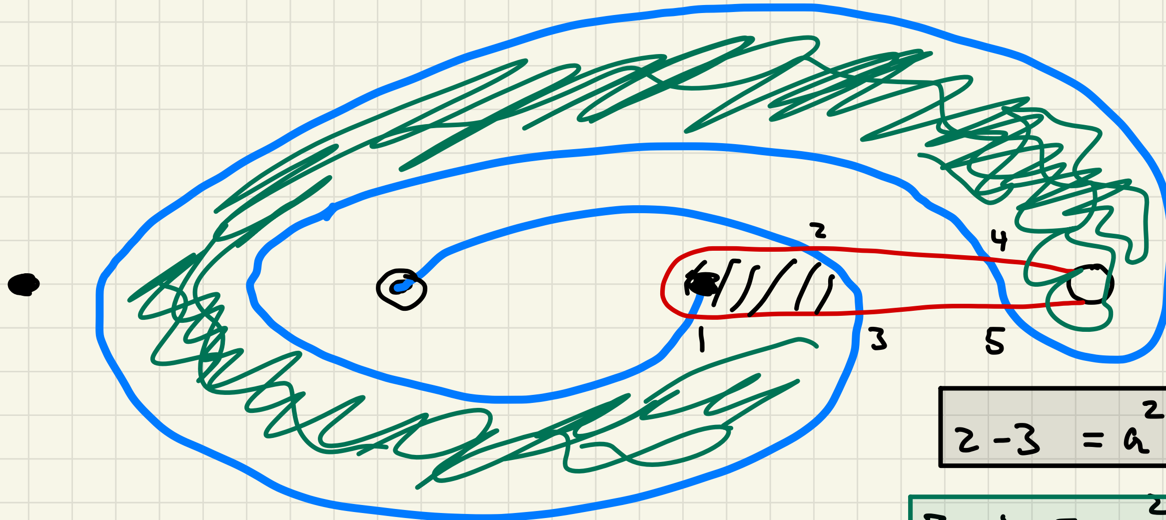
If $\varphi \in \pi_2(x,y)$

$$a(x) - a(y) = \sum_{i=1}^{3} wt^{a}_{p_i} \, n_{p_i}(\varphi)$$

$$q(x) - q(y) = \sum_{i=1}^{3} wt^{q}_{p_i} \, n_{p_i}(\varphi)$$



Ex: $P(K_{5|2})$



$\boxed{2-3 = a^2 q^{-2}}$ $\quad 4-5 = a^2 q^{-2}$

$\boxed{3-1 = q^2}$ $\quad 1-4 = q^2$

2
4 1 3
5

Other formulation:

$\boxed{\text{Bigelow}}$

$\mathrm{wt}: \pi_1(D^2-P) \to \mathbb{Z}^2$

$\rightsquigarrow$ covering space $\widetilde{D^2-P} \supset \hat{L}_{p|q}, \hat{M}_{0|1}$

$\downarrow$

$D^2-P \supset L_{p|q}, M_{0|1}$

Deck group $= \langle a, q \rangle$

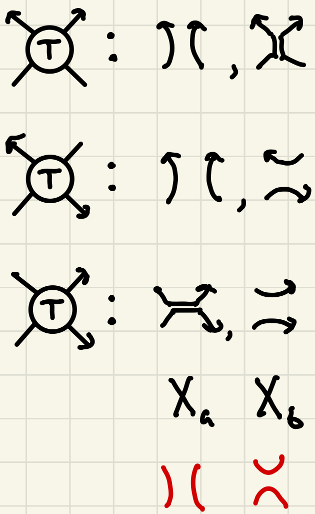
$P(K_{p|q}) \sim \hat{L}_{p|q} \cdot \hat{M}_{0|1}$

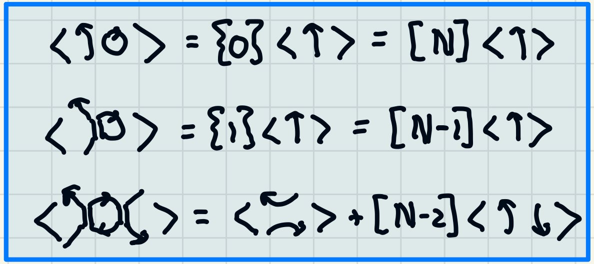
sl_N invariants of $T_{p/q}$:

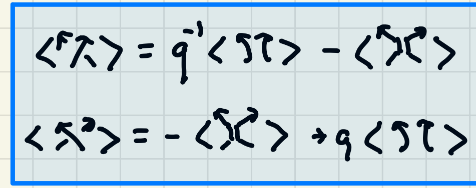
$a = q^N$

$\langle T_{p/q} \rangle \in \mathcal{W}(V, V, V^*, V^*)$

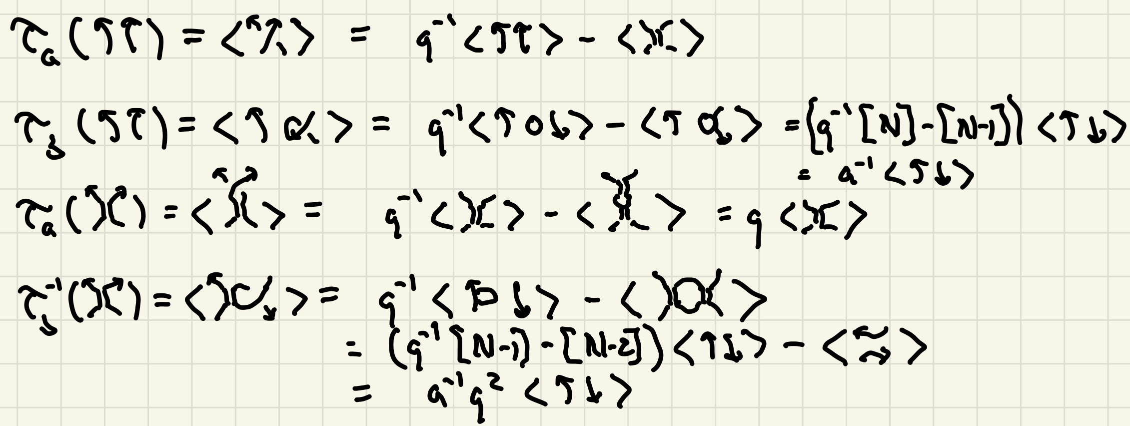
Natural basis depends on orientations:



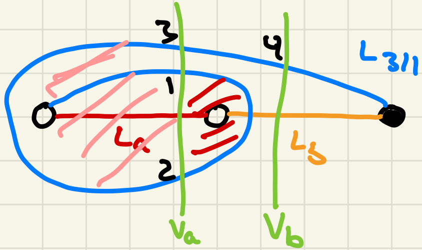


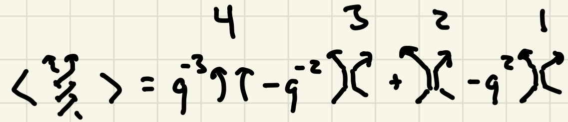


Action of τ_a, τ_b^{-1}:



Ex:





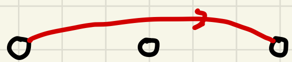
Prop: $\langle T_{p/q} \rangle \sim \hat{V}_a \cdot \hat{L}_{p/q} X_a + \hat{V}_b \cdot \hat{L}_{p/q} X_b = [\hat{L}_{p/q}]$

$X_a \longrightarrow \hat{L}_a$

$X_b \longrightarrow \hat{L}_b$

$\tau_a(L_a) =$ 

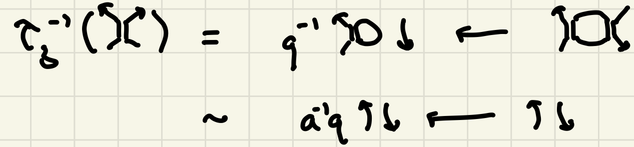
$\tau_a(L_b) =$ 

$\tau_b^{-1}(L_a) =$ 

$\tau_b^{-1}(L_b) =$ 

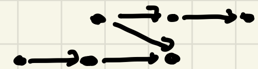
Categorification:

Action of τ_a, τ_b^{-1} is similar:



etc.

- $C(T_{p/q})$ is <u>thin</u> (linear relations between a, q, t for each type)
- $C(T_{p/q})$ is a chain / zig-zag

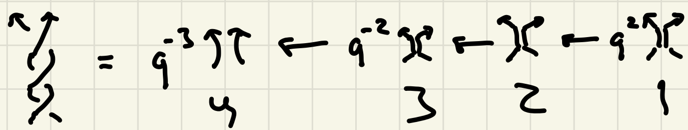


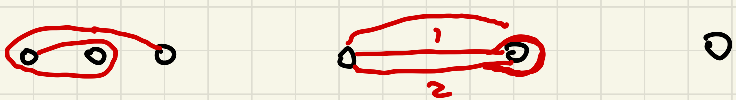
- $C(T_{p/q})$ represents $L_{p/q} \in \mathrm{Fuk}(S^2 - P)$

B. Thompson

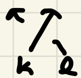
Kotelskiy
Watson
Zibrowius

Ex:



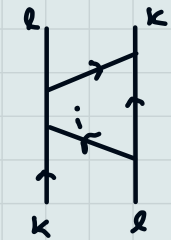


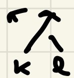
Λ^k-colored HOMFLY-PT:



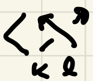
$\in \mathrm{Hom}(\tilde{\Lambda}^k \otimes \tilde{\Lambda}^l, \tilde{\Lambda}^l \otimes \tilde{\Lambda}^k) = X(k,l)$

Basis for $\underset{k \geq l}{X(k,l)} = \langle W^i_{k,l} \mid 0 \leq i \leq l \rangle$

$W^i_{k,l} =$ 

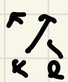
$\langle$ 

$\rangle = \sum_{i=0}^{l} (-q)^{i-l}\, W^i_{k,l}$

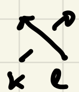
$\langle$ 

$\rangle = \sum (-q)^{l-i}\, W^i_{k,l}$

Categorification:



$: \quad q^{-l} W^0_{k,l} \leftarrow q^{-l+1} W^1_{k,l} \leftarrow \cdots \leftarrow q^0 W^l_{k,l}$



$: \quad W^l_{k,l} \leftarrow q\, W^{l-1}_{k,l} \leftarrow \cdots \leftarrow q^0 W^0_{k,l}$

Many equivalent categorifications

Cautis-Kamnitzer, Khovanov-Lauda
Stroppel, Brundan, Wu,
Webster ...

Exponential Growth:

$\mathcal{P}^{\Lambda^k}(K)$ = Poincaré polynomial of $\bar{H}^{\Lambda^k}(K)$

Conj (Gukov + Stosic): $\mathcal{P}^{\Lambda^k}(K_{p/q})\big|_{q=1} = [\mathcal{P}^{\Lambda^1}(K_{p/q})]^k\big|_{q=1}$

Thm: (Wedrich) $\dim \bar{H}^{\Lambda^k}(K) \geq (\dim \bar{H}(K))^k$

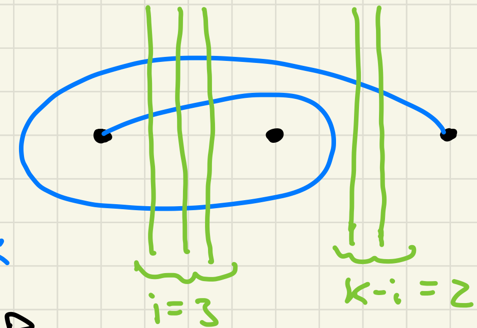
spectral sequence $\bar{H}^{\Lambda^k}(K) \rightsquigarrow (\bar{H}(K))^{\otimes k}$

$\Rightarrow \dim \bar{H}^{\Lambda^k}(K_{p/q}) \geq p^k$

Deformations: E.S. Lee
Gornik

Intersection Theory Model:

$\boxed{\text{Wedrich}}$



In $\mathrm{Sym}^k(S^2 - P_0)$ have:

Lagrangians: $\mathbb{L}_{p/q} = \mathrm{Sym}^k L_{p/q}$

$V_i = V_a^1 \times \dots \times V_a^i \times V_b^1 \times \dots \times V_b^{k-i}$ $\quad 0 \le i \le k$

Divisors: $D_i = P_i \times \mathrm{Sym}^{k-1}(S^2 - P_0)$ weights as P_i

$\Delta = \{\{z_1, \dots z_k\} \mid z_i = z_j, \text{ some } i \neq j\}$ = big diagonal $\quad wt = q^2$

$\pi_1(\mathrm{Sym}^k - \cup D_i - \Delta) \to \mathbb{Z}^2$ given by weights

+ associated covering space

Thm (Wedrich): $\langle T_{p/q} \rangle^{\wedge^k} = \sum_{i=0}^{k} (\hat{V}_i \cdot \hat{\mathbb{L}}_{p/q})\, w_i = [\hat{\mathbb{L}}_{p/q}]$

Knots-Quivers Correspondence:

Kucharski
Reinecke
Stosic
Sulkowski

Conj: Given K, there are:

① $V(K) \simeq \mathbb{Z}^{p(K)}$

② Linear forms $A, R, S: V \to \mathbb{Z}$

③ Quadratic form $Q: V \to \mathbb{Z}$

s.t.

$$\sum_{k \geq 0} P^{\Lambda^k}(K) x^k = \sum_{d \in V_{\geq 0}} (-1)^{R(d)} a^{A(d)} q^{S(d)} q^{Q(d)} \begin{bmatrix} d_1 + d_2 + \dots + d_p \\ d_1, \dots, d_p \end{bmatrix} x^{d_1 + \dots + d_p}$$

Thm (Wedrich-Stosic): True if K is arborescent.