Hilbert Schemes

+

Colors

Last time: $G = (\mathbb{C}^\times)^n \circlearrowright X$ w/ isolated fixed pts

If $G \circlearrowright V$, $|V| \in K_G(pt) = \mathrm{Rep}(G) = \mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$

Ex: If $V = \mathbb{C}$, $(z_1, \dots, z_n) \cdot w = z_1^{k_1} \cdots z_n^{k_n} w$

$$|V| = T_1^{k_1} \cdots T_n^{k_n}$$

In general, $V = \bigoplus_{i=1}^{n} V_i$ $\dim V_i = 1$

$$|V| = \sum_{i=1}^{n} |V_i|$$

Holomorphic Euler characteristic: $\mathcal{F} \in \mathrm{Coh}_G(X)$

$$\chi_G(\mathcal{F}) = \sum (-1)^i |H^i(\mathcal{F})| \in \mathrm{Rep}(G) = K_G(pt)$$

$\chi_G : K_G(X) \to \mathrm{Rep}(G)$ is an additive homomorphism

Ex: $\mathbb{C}^*$ acts on $\mathbb{P}^1$: $z\cdot[x:y] = [zx:y]$

$$H_0(\mathcal{O}(r)) = \langle x^r, x^{r-1}y, \dots, y^r \rangle \Rightarrow \chi_{\mathbb{C}^*}(\mathcal{O}(r)) = 1 + T + T^2 + \dots + T^r$$
$$= \frac{T^{r+1}-1}{T-1}$$

Localization $\Rightarrow [F] = \sum_{p \in X^G} |F_p| \, [p_*(\mathbb{C})]$

$$\Rightarrow \chi_G(F) = \sum_{p \in X^G} |F_p| \, \chi_G(p_*(\mathbb{C}))$$

Ex: If $L \in Coh_G(X)$ is a line bundle, $|L_p|$ is a monomial in $Rep(G)$

$$|(F \otimes L^r)_p| = |L_p|^r \, |F_p)$$

$$\Rightarrow \chi_G(L^r \otimes F) = \sum_{p \in X^G} |L_p|^r \alpha_p(F)$$

Hilbert Scheme of $\mathbb{C}^2$:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{ I \subset \overset{\cong \mathcal{O}_{\mathbb{C}^2}}{\mathbb{C}[x,y]} \text{ ideal} \mid \dim \mathbb{C}[x,y]/I = n \}$$

Properties:

① Smooth variety of $\dim^n$ $2n$

$$\begin{aligned} \pi : \mathrm{Hilb}^n(\mathbb{C}^2) &\to \mathrm{Sym}^n \mathbb{C}^2 \\ I &\longrightarrow \mathrm{Supp}\, I \end{aligned}$$

② $\mathrm{Hilb}^n(\mathbb{C}^2) = \mathbb{C}^2 \times \overline{\mathrm{Hilb}}^n(\mathbb{C}^2)$

$p : \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathbb{C}^2$ given by

$$\begin{aligned} \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n \mathbb{C}^2 &\to \mathbb{C}^2 \\ \{v_i\} &\longrightarrow \Sigma v_i \end{aligned}$$

$\overline{\mathrm{Hilb}}^2(\mathbb{C}^2) =$ total space of $\underset{\mathbb{P}^1}{\underset{\downarrow}{\mathcal{O}(-2)}}$

$= \bigcirc^{-2}$ (Kirby diagram)

③ Stratified by $\mu \vdash n$

deepest stratum $\overline{\mathrm{Hilb}}^n_0$

is compact of $\dim^n$ $n-1$

Ex: $n=2$ $\overline{\mathrm{Hilb}}^2_0 = \mathbb{P}^1$

$I \in \overline{\mathrm{Hilb}}^2_0 \iff I = \langle ax+by, x^2, xy, y^2 \rangle$

$$\begin{aligned} \overline{\mathrm{Sym}}^2 \mathbb{C}^2 &= \{ \{v,w\} \mid v+w=0 \} \\ &= \{ \{v,-v\} \mid \vec{v} \in \mathbb{C}^2 \} \\ &= \mathbb{C}^2/\sim \quad v \sim -v \\ &= \mathrm{Cone}(L(2,1)) \end{aligned}$$

④ Tautological Bundle

$$\begin{tikzcd} \mathbb{C}[x,y]/I \arrow[r] & \tau \arrow[d] \\ & \mathrm{Hilb}^n \end{tikzcd}$$

n-dim'l vector bundle

$\tau = \bar{\tau} \oplus \mathbb{C}$

$$\begin{tikzcd} \bar{\tau}\big|_{\overline{\mathrm{Hilb}}^2_0} = \mathcal{O}(-1) \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

Thm (Haiman): $\mathcal{O}(1) := \det(\tau^\vee)$ is <u>ample</u>.

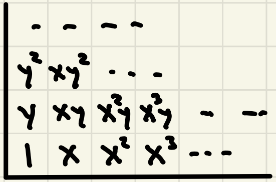
$H^*(F \otimes \mathcal{O}(n)) = 0$ for $* > 0$ and $n \gg 0$

⑤ $(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}[x,y]$

$\Rightarrow (\mathbb{C}^*)^2 \curvearrowright \mathrm{Hilb}^n, \overline{\mathrm{Hilb}^n}$

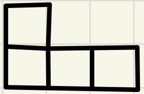
$(z,w) \cdot p(x,y) = p(zx, wy)$

Fixed pts ⟷ monomial ideals ⟷ partitions $\lambda \vdash n$

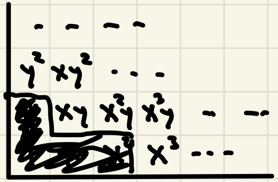


$I_\lambda = \langle \text{monomials} \notin \lambda \rangle$

$\lambda =$



$I_\lambda =$



Dream:

Realized by Oblomkov-Rozansky

$\sigma \in Br_n$, $\tilde{\sigma} \subset S^1 \times D^2$ = closure

$$\tilde{\sigma} \subset S^1 \times D^2 \longrightarrow F(\tilde{\sigma}) \in D^b Coh_G \overline{Hilb}^n$$

$$\overline{H}(\bar{\sigma}) \longrightarrow H^*_G(F(\sigma) \otimes \Lambda^* \overline{\mathcal{T}})$$

$a \longleftrightarrow$ grading in Λ^*

$q,t \longleftrightarrow T_1, T_2$, grading in H^*

Adding a twist $\longrightarrow$ $F(\widetilde{\sigma Tw_n}) = F(\sigma) \otimes \mathcal{O}(1)$

$T(1,n) \longrightarrow F(\sigma_1 \dots \sigma_{n-1}) = \mathcal{O}_{\overline{Hilb}^n_0}$

$\tilde{\sigma}$ is a knot $\longrightarrow$ $Supp\ F(\sigma) \subset \overline{Hilb}^n_0$

Consequences:

1) Formula for Twisting

$F(K_r) = F(\sigma)\otimes\mathcal{O}(r)$

$\mathcal{O}(1)$ ample $\Rightarrow H^i(F(\sigma)\otimes\mathcal{O}(r))=0$
for $r\gg 0$, $i\neq 0$

For $r\gg 0$, $\overline{H}(K_r) = H^0(F(\sigma)\otimes\mathcal{O}(r))$

$$\mathcal{P}(\overline{H}(K_r)) = \chi_G(F(\sigma)\otimes\mathcal{O}(r))$$

$$= \sum_{\lambda\vdash n} T_1^{r\kappa(\lambda)}T_2^{r\kappa(\lambda')}\varphi_\lambda(\sigma)$$

Elias, Hogancamp, Gorsky,
Mellit, Negut...

$$H^{top} = H^*(F\otimes \underset{\mathcal{O}(1)}{\underset{\parallel}{\Lambda^{top}\mathcal{T}^\vee}})$$

2) Symmetry: $\bar{\sigma}$ a knot $\Rightarrow F(\sigma)\in D^b\,Coh_{GL_2}(\overline{Hilb}^n)$
$\leftrightarrow$ symmetry generalizing $P_K(a,q) = P_K(a,q^{-1})$

3) $H_*^{bot}(\bar{\sigma}) \simeq H_*^{top}(\overline{\sigma Tw_n})$ categorifies thm of Kalman

Decategorification:

$$\begin{tikzcd}
\sigma \in Br_n \arrow[r] \arrow[d] & \tilde{\sigma} \subset S^1 \times D^2 \arrow[d] \\
\psi(\sigma) \in H_n \arrow[r, "\pi"] & \hat{\psi}(\sigma) \in H_n / [H_n, H_n] =: Z_n
\end{tikzcd}$$

$\tilde{\sigma}' = \tilde{\sigma} \iff \sigma' = \tau \sigma \tau^{-1}$

$$= \langle z_\lambda \mid \lambda \vdash n \rangle$$

$$z_\lambda = \pi(e_\lambda)$$

$$H_n \simeq \bigoplus_\lambda M_{n_\lambda \times n_\lambda}(\mathbb{C}(q))$$

$$\mathfrak{gl}_n / [\mathfrak{gl}_n, \mathfrak{gl}_n] \simeq \mathbb{C}$$

$$X \longmapsto \operatorname{tr} X$$

$$\sigma \longrightarrow [F(\tilde{\sigma})] \in K(\overline{\mathrm{Hilb}}^n)$$

$$= \langle P_{\lambda *}(\mathbb{C}) \mid \lambda \vdash n \rangle$$

$$K(\overline{\mathrm{Hilb}}^n) \simeq Z_n$$

$$P_{\lambda *}(\mathbb{C}) \longmapsto z_\lambda$$

$$[F(\tilde{\sigma})] \to \hat{\psi}(\sigma)$$

(Braid like) Satellites:

$\sigma \in Br_n$

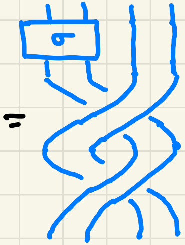
$\tilde{\sigma} \subset S^1 \times D^2 , K \subset S^3 \longrightarrow$ Satellite $K(\tilde{\sigma}) \subset S^3$

Prop: $P(K(\tilde{\sigma})) = \hat{P}_{K,n}(\hat{\varphi}(\sigma))$ $\quad \hat{P}_{K,n} \in Z_n^*$

Proof: $\sigma \in Br_n, \ \tau \in Br_m \longrightarrow C_\tau(\sigma) \in Br_{nm}$

$\bar{\tau} = K \Rightarrow \overline{C_\tau(\sigma)} = K(\tilde{\sigma})$

$\tau =$ 

$C_\tau(\sigma) =$ 

$$\begin{tikzcd}
Br_n \arrow[r, "C_\tau"] \arrow[d] & Br_{nm} \arrow[d] \\
H_n \arrow[r, "C_\tau"] \arrow[d, "\pi"] & H_{nm} \arrow[d, "\pi"] \\
Z_n \arrow[r] & Z_{nm}
\end{tikzcd}$$

$$\widetilde{C_\tau(\sigma\sigma')} = \widetilde{C_\tau(\sigma'\sigma)}$$

$$\Downarrow$$

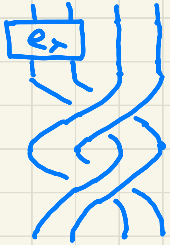
$$\pi(C_\tau(ax)) = \pi(C_\tau(xa))$$

Colors:

Def: $P^{\lambda}(K) = \hat{P}_{K,n}(z_{\lambda}) = P(K(e_{\lambda}))$

is the λ-colored HOMFLY-PT

polynomial of K.



If $\hat{\Psi}(\sigma) = \sum c_{\lambda} z_{\lambda}$

$$P(K(\tilde{\sigma})) = \sum c_{\lambda} P^{\lambda}(K)$$

Categorification?

λ' = transpose of λ

Symmetry: $P^{\lambda}(K)\big|_{q \to q^{-1}} = P^{\lambda'}(K)$

Witten-Reshetikhin-Turaev:

$\mathfrak{g}$ = simple Lie algebra ($\mathfrak{g} = \mathfrak{sl}_n$)

Category $\mathrm{Tan}_{\mathfrak{g}}$

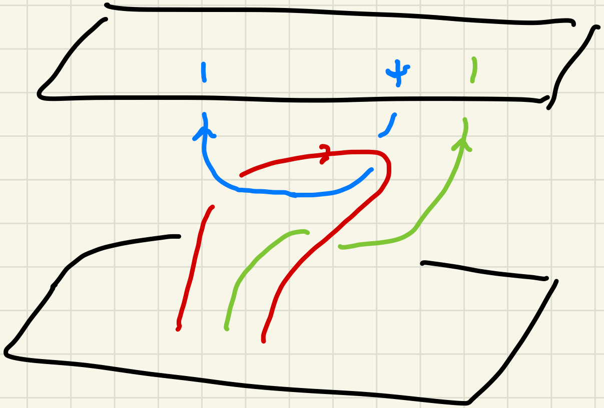
objects: oriented codimension 2 submanifolds of $\mathbb{R}^2$ labeled by elements of $\mathrm{Rep}(\mathfrak{g})$

morphisms: colored oriented tangles

$$\mathcal{W}_{\mathfrak{g}} : \mathrm{Tan}_{\mathfrak{g}} \longrightarrow U_q(\mathfrak{g})\text{-mod}$$

monoidal functor

$$\mathcal{W}_{\mathfrak{g}}(X \amalg Y) = \mathcal{W}_{\mathfrak{g}}(X) \otimes \mathcal{W}_{\mathfrak{g}}(Y)$$



Bijection: $\mathrm{Rep}(\mathfrak{g}) \longleftrightarrow \mathrm{Rep}(U_q(\mathfrak{g}))$

$V \longleftrightarrow \tilde{V}$

$\mathbb{C}$ $\quad$ $R = \mathbb{Z}[q^{\pm 1}]$

$T : (V_1, \dots V_k) \to (V_1', \dots V_\ell')$

$\rightsquigarrow \langle T \rangle \in \mathrm{Hom}(\tilde{V}_1 \otimes \dots \otimes \tilde{V}_k, \tilde{V}_1' \otimes \dots \otimes \tilde{V}_\ell')$

$g = sl_N$:

$\mathrm{Rep}(sl_N) = \langle V_\lambda \mid \lambda \vdash k,\ k \geq 0,\ \text{all } \lambda_i < N \rangle$

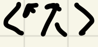
Ex: $T =$  $\langle T \rangle \in \mathrm{End}(\tilde{V} \otimes \tilde{V})$

$\simeq \mathrm{End}(V \otimes V) \otimes R$

$V \otimes V \simeq \Lambda^2 V \oplus \mathrm{Sym}^2 V$

$= V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus V_{\square\square}$

$= \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right\} \simeq R^2$

$\langle$  $\rangle$, $\langle$  $\rangle$, $\langle \uparrow\uparrow \rangle \in \mathrm{End}(\tilde{V} \otimes \tilde{V}) \simeq R^2$

$\Rightarrow$ linearly dependent $\Rightarrow$ skein relation

In $\mathrm{End}(\tilde{V} \otimes \tilde{V}) = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$

$\uparrow\uparrow \quad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

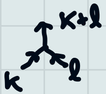
 $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$V = V_{\square} = \mathbb{C}^n$

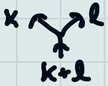
$V_{\underbrace{\square\square\square}_{k}} = \mathrm{Sym}^k V$

$V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}k} = \Lambda^k(V)$

MOY notation: $|_k = |\ \Lambda^k(V)$

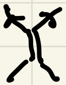
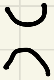


$= \Lambda^k(V) \otimes \Lambda^l(V) \longrightarrow \Lambda^{k+l}(V)$

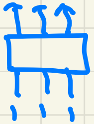


$= \Lambda^{k+l}(V) \longrightarrow \Lambda^k(V) \otimes \Lambda^l(V)$

$N = 2$: $\Lambda^2 V = \mathbb{C} = V_\emptyset$

 $=$ 

Hecke algebra again:

$T =$  $\langle T \rangle \in \mathrm{End}(\tilde{V}^{\otimes n})$

Schur-Weyl duality

$\mathfrak{sl}_N, S_n \circlearrowright V^{\otimes n} \simeq \bigoplus_{\lambda \vdash n} V_\lambda \otimes S_\lambda$

$\Rightarrow \mathrm{End}_{\mathfrak{sl}_n}(V^{\otimes n}) \simeq \bigoplus_\lambda M_{n_\lambda \times n_\lambda} \simeq \mathbb{C}[S_n]$

for $N > n$

quantum S-W duality:

$U_q(\mathfrak{sl}_n), \mathcal{H}_n \circlearrowright \tilde{V}^{\otimes n} \simeq \bigoplus_\lambda \tilde{V}_\lambda \otimes \tilde{S}_\lambda$

$\mathrm{End}_{U_q(\mathfrak{sl}_n)}(\tilde{V}^{\otimes n}) \simeq \mathcal{H}_n$

HOMFLY-PT: fix T, vary N

For $n \gg 0$, $\mathrm{End}_{\mathfrak{sl}_N}(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_k}, V_{\mu_1} \otimes \dots \otimes V_{\mu_\ell}) = X_N$

stabilizes, $\langle T \rangle \in X \otimes \mathbb{Z}[a^{\pm 1}]$

$\langle T \rangle_{\mathfrak{sl}_N} = \langle T \rangle |_{a=q^N}$

$\sigma \in \mathfrak{Br}_n$

$\langle \sigma \rangle = \psi(\sigma)$