Twisting, Projectors + Hilbert Scheme of $\mathbb{C}^2$

Rougier Complex $C(\sigma)$

$\sigma \in Br_n$

is a complex over the category $\mathcal{C} = SBim_n$

$$\cdots \leftarrow C_{i-1} \xleftarrow{d_i} C_i \xleftarrow{d_{i+1}} C_{i+1} \leftarrow \cdots$$

Can't take homology, but chain hty, chain hty $\sim$ make sense.

$C_i \in Obj(\mathcal{C})$, $d_i \in Hom(C_i, C_{i-1})$

$d^2 = 0$

Compare w/ $\widehat{CFD}(Y)$, which is a twisted complex over diff'l graded category $A(\partial Y)$

Objects $\leftrightarrow$ idempotents

Morphisms $\leftrightarrow$ algebra elements

(Shifted) Rouquier Complex:

$$\tilde{C}(\sigma_i) = \overset{0}{\boxed{R}} \xleftarrow{x_2} \overset{1}{qB_1}$$

$$\tilde{C}(\sigma_i^{-1}) = \boxed{q^{-1}B_1} \xleftarrow{x_1} R$$

$$\tilde{C}(\sigma\tau) = \tilde{C}(\sigma) \otimes \tilde{C}(\tau)$$

$\mathrm{Hom}(1,1) \simeq R\cdot\mathrm{id}_1$ $\qquad$ $\mathrm{Hom}(\mathbb{B},1) \simeq R\cdot X_2$

$\mathrm{Hom}(1,\mathbb{B}_1) \simeq R\cdot X_1$ $\qquad$ $\mathrm{Hom}(\mathbb{B},\mathbb{B}) \simeq \mathbb{B}\cdot\mathrm{id}_{\mathbb{B}}$

Gradings: $q(\mathrm{id}_R) = q(\mathrm{id}_{B_1}) = 0$

$q(x_1) = q(x_2) = 1$ $\qquad$ $q(f\circ g) = q(f) + q(g)$

$q(X_i) = q(X_i') = 2$

If $\varphi: q^a X \to q^b Y$, $q(\varphi) = a-b$

Homfly-PT homology:
(Khovanov-Rozansky)

$$\overline{H}(\bar{\sigma}) = (aq^{-1})^{w(\sigma)} H_*(\overline{HH}_*(\tilde{C}(\sigma)))$$

Ex: $n=2$ $\qquad$ $\overline{Tr}(1) = \{0\} = \dfrac{a-a^{-1}}{q-q^{-1}}$

$$\overline{HH}_*(R) = \oplus \begin{array}{l} aq\,\mathbb{C}[x] \\ a^{-1}q\,\mathbb{C}[x] \end{array} \qquad \overline{HH}_*(B_1) = \oplus \begin{array}{l} a\,\mathbb{C}[x] \\ a^{-1}q^2\,\mathbb{C}[x] \end{array}$$

$$\overline{HH}_*(\tilde{C}_*(\sigma_1)) = \begin{array}{ccc} aq\,\mathbb{C}[x] & \xleftarrow{\cdot 1} & aq\,\mathbb{C}[x] \\ a^{-1}q\,\mathbb{C}[x] & \xleftarrow{\cdot x} & a^{-1}q^3\,\mathbb{C}[x] \end{array}$$

$$\Rightarrow \overline{H}_*(\bar{\sigma}_1) = aq^{-1}\begin{bmatrix} 0 & 0 \\ \boxed{a^{-1}q\cdot 1} & 0 \end{bmatrix} = \mathcal{Z} = \overline{H}_*(0)$$

Let's compute $\tilde{C}(\sigma_1^n)$

Filtration Trick: If C is a filtered cx w/ associated graded cx's $C^{(i)}$, then $C \sim C'$, where C' is filtered w/ associated gradeds $M(C^{(i)})$

$\uparrow$ Minimal cx

Proof: Cancel!

Claim: $\tilde{C}(\sigma_1^n) = R \leftarrow qB_1 \leftarrow q^3 B_1 \leftarrow \dots \leftarrow q^{2n-1} B_1$

Basic Lemma:

$$\boxed{B_1 \otimes \tilde{C}(\sigma_1) = q^2 t B_1}$$

Proof: Induct

$$\begin{tikzcd}
 & qB_1 \arrow[d] & q^2B_1^2 \arrow[l] \arrow[d] & q^4B_1^2 \arrow[l] \arrow[d] & \dots \arrow[l] & q^{2n}B_1^2 \arrow[l] \arrow[d] \\
\tilde{C}(\sigma_1^n) \otimes \tilde{C}(\sigma_1) = & R & qB_1 \arrow[l] & q^3B_1 \arrow[l] & \dots \arrow[l] & q^{2n-1}B_1 \arrow[l]
\end{tikzcd}$$

$$\rotatebox{90}{$=$}$$

$\tilde{C}(\sigma_1^{n+1})$

$$\sim \begin{tikzcd}
qB_1 \arrow[d] & q^3B_1 \arrow[l] & q^5B_1 \arrow[l] & \dots \arrow[l] & q^{2n+1}B_1 \arrow[l] \\
R
\end{tikzcd}$$

$(X_1'-X_1)(X_1'-X_2)=0$

in $\mathrm{End}(B_1)$

$d^2=0$ and gradings $\Rightarrow$ diff's are $R \xleftarrow{x_2} qB_1 \xleftarrow{x_1'-x_1} q^3B_1 \xleftarrow{x_1'-x_2} q^5B_1 \xleftarrow{x_1'-x_1} q^7B_1 \dots$

$\tilde{C}(\sigma) \otimes \tilde{C}(\sigma^{-1}) \sim \tilde{C}(\mathrm{id}) \simeq R \Rightarrow \tilde{C}(\sigma) \not\simeq C \oplus C'$

minimal cx

Stable Limits:

Def: $C^{(i)}$ a sequence in $\mathrm{Kom}(SBim_n)$

Say $\lim\limits_{i\to\infty} C^{(i)} = C$ if $\forall n\ \exists N$ s.t.

$C^{(i)}_{*<n} \simeq C_{*<n}$ for all $i > N$.

Lemma: If $f_i : C_i \to C_{i+1}$ satisfies $\lim\limits_{i\to\infty} \mathrm{Cone}(f_i) = 0$, then $\lim\limits_{i\to\infty} C^{(i)}$ exists.

Thm (Stosic): $\lim\limits_{i\to\infty} \tilde{C}(Tw_n^i)$ exists.

Def: If $I \subset H_n$ is an ideal

$$X_k(I) = \{ C \in \mathrm{Kom}(I) \mid C_i = 0 \text{ for } i<k \}$$

$I_{max} = \langle B_s \mid s \neq e \rangle$

1) $C \in X_k(I),\ C' \in X_\ell(I') \Rightarrow C \otimes C' \in X_{k+\ell}(I \cap I')$

2) $\tilde{C}(\sigma_i) \cdot X_k(I) \subset X_k(I)$ $\quad\quad \tilde{C}(\sigma_i) \in X_0(\mathcal{H}_n)$

3) If σ is a positive braid $\quad \tilde{C}(\sigma) \cdot X_k(I) \subset X_k(I)$

4) $\widetilde{C}(\sigma_i) \otimes B_i \in X_1(I_{max})$ (Basic Lemma)

5) $\widetilde{C}(Tw_n) \otimes B_i \in X_1(I_{max})$ (write $Tw_n = \sigma \sigma_i$, $\sigma \in Br_n^+$)

6) If $B \in I_{max}$, $\widetilde{C}(Tw_n) \otimes B \in X_1(I_{max})$ $B = \sum B_i y^i$

7) $\widetilde{C}(Tw_n) \cdot X_i(I_{max}) \subset X_{i+1}(I_{max})$ (Filtration Trick)

Proof of Thm: $Tw_n \in Br_n^+$, so have $p: \widetilde{C}(id) \to \widetilde{C}(Tw_n)$

$Cone(p) \in X_0(I_{max})$

$f_i = id \otimes p: \widetilde{C}(Tw_n^i) \to \widetilde{C}(Tw_n^{i+1})$ $\Rightarrow \lim_{i \to \infty} Cone(f_i) = 0$

$Cone(f_i) = \widetilde{C}(Tw_n^i) \otimes Cone(p) \in X_i(I_{max})$ by ⑦ $\square$

Thm (Rozansky): $E = \lim_{i \to \infty} \widetilde{C}(Tw_n^i)$ categorifies $e_{\underbrace{\boxed{}\boxed{}\cdots\boxed{}}_{n}}$

in the sense that 1) $E \otimes E = E$

2) $E \cdot I_{max} = 0$ (follows from ⑦)

$n=2$:

$$\begin{tikzcd} E_{\ydiagram{2}} \arrow[d, "p"] & = & R & q B_1 \arrow[l, "x_2"', "/" marking] & q^3 B_1 \arrow[l, "x_1-x_1'"'] & q^5 B_1 \arrow[l, "x_2-x_1'"'] & q^7 B_1 \arrow[l, "x_1-x_1'"'] & \ldots \arrow[l] \\ E_{\ydiagram{1,1}} & = & & q B_1 & q^3 B_1 \arrow[l, "x_1-x_1'"'] & q^5 B_1 \arrow[l, "x_2-x_1'"'] & q^7 B_1 \arrow[l, "x_1-x_1'"'] & \ldots \arrow[l] \end{tikzcd}$$

$e_{\ydiagram{2}} + e_{\ydiagram{1,1}} = 1$

categorify $e_{\ydiagram{2}}, e_{\ydiagram{1,1}}$: $\chi(E_\lambda) = e_\lambda$ $\quad E_{\ydiagram{2}} \otimes E_{\ydiagram{1,1}} = 0$, $E_{\ydiagram{1,1}} \otimes E_{\ydiagram{1,1}} = E_{\ydiagram{1,1}}$ etc.

$$\tilde{C}(Tw_2) \otimes E_{\ydiagram{2}} = E_{\ydiagram{2}}$$

$$\tilde{C}(Tw_2) \otimes E_{\ydiagram{1,1}} = q^4 t^2 E_{\ydiagram{1,1}}$$

$e_{\ydiagram{1}}$

$$R = \mathrm{Cone}(p) \Rightarrow \tilde{C}(\sigma_1) = \left[\tilde{C}(\sigma_1) \otimes E_{\ydiagram{2}} \xrightarrow{p} \tilde{C}(\sigma_1) \otimes E_{\ydiagram{1,1}}\right]$$

$$= E_{\ydiagram{2}} \longrightarrow \tilde{E}_{\ydiagram{1,1}}$$

$$\tilde{C}(Tw_2^s \sigma_1) = \left[E_{\ydiagram{2}} \to q^{4s} t^{2s} \tilde{E}_{\ydiagram{1,1}}\right]$$

$$\begin{tikzcd} \tilde{E}_{\ydiagram{1,1}} & = & q^3 B_1 & q^5 B_1 \arrow[l, "x_2-x_1'"'] & q^7 B_1 \arrow[l, "x_1-x_1'"'] & \ldots \arrow[l] \end{tikzcd}$$

$\rightsquigarrow$ Formula for $H_*(K_r) = H_*(T(2,2r+1))$

$$\overline{HH}_*(E_{\square\square}) = \begin{matrix} aq\,\mathbb{C}[x] \xleftarrow{\cdot 1} aq\,\mathbb{C}[x] \xleftarrow{\cdot 0} aq^3\mathbb{C}[x] \xleftarrow{\cdot x} aq^5\mathbb{C}[x] \cdots \\ a^{-1}q\,\mathbb{C}[x] \xleftarrow{\cdot x} a^{-1}q^3\mathbb{C}[x] \xleftarrow{\cdot 0} a^{-1}q^5\mathbb{C}[x] \xleftarrow{\cdot x} a^{-1}q^7\mathbb{C}[x] \cdots \end{matrix}$$

$$H_*(\overline{HH}_*(E_{\square\square})) = a^{-1}q\ \mathbb{C}[z] \otimes \Lambda^*(\theta_1) \qquad |z| = q^4t^2 \quad |\theta| = a^2q^2t^2$$

$$\overline{HH}_*(\widetilde{E}_{\substack{\square\\\square}}) = \begin{matrix} aq^3\mathbb{C}[x] \xleftarrow{\cdot x} aq^5\mathbb{C}[x] \xleftarrow{0} aq^7\mathbb{C}[x] \xleftarrow{\cdot x} aq^9\mathbb{C}[x] \cdots \\ a^{-1}q^5\mathbb{C}[x] \xleftarrow{\cdot x} a^{-1}q^7\mathbb{C}[x] \xleftarrow{0} a^{-1}q^9\mathbb{C}[x] \xleftarrow{\cdot x} a^{-1}q^{11}\mathbb{C}[x] \cdots \end{matrix}$$

$$H_*(\overline{HH}_*(\widetilde{E}_{\substack{\square\\\square}})) = a^{-1}q^5t^2\ \mathbb{C}[z] \otimes \Lambda^*(\theta') \qquad |z| = q^4t^2 \quad |\theta'| = a^2q^{-2}$$

$$\Rightarrow \mathcal{P}(K_r) = (aq^{-1})^{2r+1}\left[\mathcal{P}(E_{\square\square}) - q^{4r}t^{2r}\,\mathcal{P}(\widetilde{E}_{\substack{\square\\\square}})\right]$$

$$\boxed{(aq^{-1})^{2r}}\left[\frac{(1+a^2q^2t^2)}{1-q^4t^2} - \boxed{q^{4(r+1)}t^{2(r+1)}}\,\frac{(1+a^2q^{-2})}{1-q^4t^2}\right]$$

Formula for $\mathcal{P}(K_r)$

r enters only in exponents of monomials

BUT: only works for $r>0$.

Compare with:

$$\begin{tikzcd} \mathcal{O}(r) \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

$$H^0(\mathcal{O}(r)) = \langle X^r, X^{r-1}Y, \dots Y^r \rangle \circlearrowleft \mathfrak{sl}_2$$

$$q\dim H^0(\mathcal{O}(r)) = [r+1] = \frac{q^r - q^{-r}}{q - q^{-1}} \quad \text{for } r>0$$

In general, $\dfrac{q^r - q^{-r}}{q - q^{-1}} = \chi_q(H^*(\mathcal{O}(r)))$

In fact, $H_*(T(2,2r+1)) = \oplus \begin{array}{l} a\, H^*(\mathcal{O}(r-1)) \\ a^{-1} H^*(\mathcal{O}(r)) \end{array}$

Equivariant Localization:

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd}$$

EG contractible

Equivariant H^*: $G \circlearrowright X$ $X \times_G EG = X \times EG / \sim$ $(x,y) \sim (gx, gy)$

$$H^*_G(X) = H^*(X \times_G EG)$$

$$\begin{tikzcd} X \times_G EG \arrow[d] \\ EG/G = BG \end{tikzcd}$$

$\Rightarrow H^*_G(X)$ is module over $H^*(BG) = H^*_G(pt)$

$G = S^1$: $H^*(BG) = H^*(\mathbb{C}P^\infty)$
$= \mathbb{Z}[U] = R$
$R_\infty = \mathbb{Z}[U^{\pm 1}]$

$X^G = \{x \in X \mid gx = x, \forall g \in G\}$ $H^*_G(X^G) = H^*(X^G \times BG) = H^*(X) \otimes H^*(BG)$

Localization: $\iota^*: H^*_{S^1}(X) \otimes_R R^\infty \xrightarrow{\sim} H^*_{S^1}(X^{S^1}) \otimes_R R^\infty$ $\iota: X^G \hookrightarrow X$

"Ex": $\widehat{HF}(Y) \longleftrightarrow H^*(SWF(Y))$ Manolescu

$HF^-(Y) \longleftrightarrow H^*_{S^1}(SWF(Y))$

$HF^\infty(Y) \longleftrightarrow H^*_{S^1}(SWF(Y)^{S^1})$ (But take care if $b_1(Y) \geq 3$)

K-Theory:

G = algebraic group

$G \circlearrowright X$ variety

Sheaf $\begin{matrix} F \\ \downarrow \\ X \end{matrix}$ is equivariant if
$$\begin{tikzcd} G \circlearrowright F \arrow[d] & \arrow[d] \\ G \circlearrowright X & \end{tikzcd}$$
commutes

$K_G(X) = \langle F \in Coh_G(X) \rangle / \sim$

$0 \to A \to B \to C \to 0 \Rightarrow A + C = B$

Ex: $K_G(pt) = Rep(G)$

$G = (\mathbb{C}^*)^n = \mathbb{C}^n / L \quad Rep(G) = \mathbb{Z}[L^\vee] \simeq \mathbb{Z}[T_1^{\pm 1}, \dots T_n^{\pm 1}]$

$K_G(X)$ is a module over $K_G(pt) = Rep(G)$

Localization: $G = (\mathbb{C}^*)^n$

$$\iota^* : K_G(X) \otimes \mathbb{Q}(T_1, \dots T_n) \xrightarrow{\simeq} K_G(X^G) \otimes \mathbb{Q}(T_1, \dots T_n)$$

If G has isolated fixed pts,

$$K_G(X) \otimes \mathbb{Q}(T_1, \dots T_n) = \langle p_*(\mathbb{C}) \mid p \in X^G \rangle$$

$p_*(\mathbb{C})$ = skyscraper sheaf w/ stalk $\mathbb{C}$ at p.

Holomorphic Euler characteristic:

$$F \in \mathrm{Coh}_G(X) \Rightarrow G \circlearrowright H^*(F)$$

$$\Rightarrow \chi_G(F) = \sum (-1)^i \, |H^i(F)| \in \mathrm{Rep}(G)$$

$$0 \to A \to B \to C \to 0$$

$$\Rightarrow \chi_G(B) = \chi_G(A) + \chi_G(C)$$

$$\Rightarrow \chi_G : K_G(X) \to \mathrm{Rep}(G)$$

Additive homomorphism

Not a ring homomorphism

Ex: $\mathbb{C}^*$ acts on $\mathbb{P}^1$: $z \cdot [x;y] = [zx:y]$

$$H_0(\mathcal{O}(r)) = \langle x^r, x^{r-1}y, \dots, y^r \rangle \Rightarrow \chi_{\mathbb{C}^*}(\mathcal{O}(r)) = 1 + T + T^2 + \dots + T^n$$

$$= \frac{T^{n+1}-1}{T-1}$$

If $G = (\mathbb{C}^*)^n$ has isolated fixed pts

$$[F] = \sum_{p \in X^G} |F_p| \, [p_*(\mathbb{C})]$$

$$\chi_G(F) = \sum_{p \in X^G} |F_p| \, \chi_G(p_*(\mathbb{C}))$$

Ex: If $L \in \mathrm{Coh}_G(X)$ is a line bundle,

$|L_p|$ is a monomial in $\mathrm{Rep}(G)$

$$\chi_G(L^r \otimes F) = \sum_{p \in X^G} |L_p|^r \alpha_p(F)$$

Dream:

Realized by Oblomkov-Rozansky

$\sigma \in Br_n$, $\tilde{\sigma} \subset S^1 \times D^2$ = closure

$$\tilde{\sigma} \subset S^1 \times D^2 \longrightarrow F(\tilde{\sigma}) \in D^b Coh_G \, \overline{Hilb}^\wedge$$

$$\overline{H}(\overline{\sigma}) \longrightarrow H^*_G(F(\sigma) \otimes \Lambda^* \overline{\mathcal{T}})$$

$a \longleftrightarrow$ grading in Λ^*

$q, t \longleftrightarrow T_1, T_2$, grading in H^*

$$\text{Adding a twist} \longrightarrow F(\widetilde{\sigma \, Tw_n}) = F(\sigma) \otimes \mathcal{O}(1)$$

$$T(1,n) \longrightarrow F(\sigma_1 \ldots \sigma_{n-1}) = \mathcal{O}_{\overline{Hilb}_0^\wedge}$$