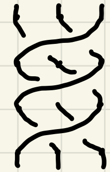
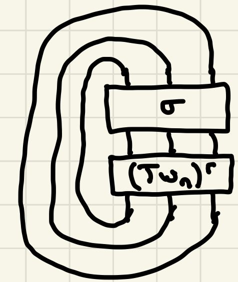
Adding Twists

Adding Twists:

$Tw_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \in Br_n$ $\qquad$ $Tw_3 =$ 

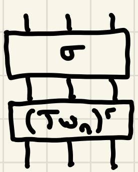
$= \text{full twist}$ $\qquad$ $K_r =$ 

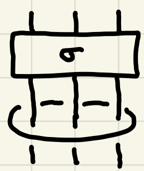
Fix $\sigma \in Br_n$, let $K_r = \overline{(Tw_n)^r \sigma}$.

Ex: If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$, $K_r = T(rn+1, n)$ torus knot

Q: How does $P(K_r)$ vary with r?

Easier: What is the Alexander polynomial $\Delta(K_r)$?

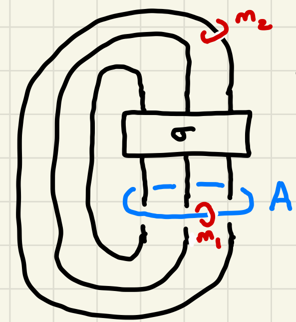


$=$


$-1/r$

2-component link

$L = \overline{\sigma} \cup A$
$\qquad\qquad\quad \parallel$
$\qquad$ Braid Axis

$L =$ 

$H_1(S^3 - L) = \langle m_1, m_2 \rangle \cong \mathbb{Z}^2$

$s = [m_1]$
$t = [m_2]$

$\ell_1 = n\, m_2$

Multivariable Δ:

$\Delta(L) \in \mathbb{Z}[H_1(S^3 - L)] = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$

Milnor Torsion $\tau(S^3 - L) = \Delta(L)$

$S^3 - K_r = (S^3 - L) \cup_{T^2} (S^1 \times D^2)$ $\qquad \tau(S^3 - K) = \dfrac{\Delta(K)}{t-1}$

(Dehn fill A along $m_1 - r\ell_1$)

Product formula for τ: $\qquad \tau(S^1 \times D^2) = \dfrac{1}{[S^1] - 1}$

$$\tau(S^3 - K_r) = c_{1*}(\tau(S^3 - L))\, c_{2*}(\tau(S^1 \times D^2))$$

$$\frac{\Delta(K_r)}{t-1} = \frac{\Delta(L)}{t^n - 1}\Big|_{s = t^{rn}}$$

If $\Delta(L) = \sum_{i=0}^{n-1} p_i(t)\, s^i$,

$$\boxed{\Delta(K_r) = \frac{t-1}{t^n - 1} \sum_{i=0}^{n} p_i(t)\, t^{irn}}$$

$$H_1(S^3 - K_r) = H_1(S^3 - L)/\langle m_1 - r\ell_1 \rangle$$
$$= \langle m_1, m_2 \rangle / \langle m_1 - rn\, m_2 \rangle$$
$$\cong \langle m_2 \rangle$$

$c_{1*}(t) = t \qquad c_{1*}(s) = t^{rn}$

$[S^1] \cdot (m_1 - r\ell_1) = 1 \Rightarrow [S^1] = \ell_1$

$\Rightarrow c_{2*}([S^1]) = [\ell_1] = t^n$

$\deg_s \Delta(L) = -\chi(S)$

Ex: $n = 2$ $\qquad K_r = T(2, 2r+1)$

$$\Delta(K_r) = \frac{t-1}{t^2 - 1}(1 + ts)\Big|_{s = t^{2r}} = \frac{t^{2r+1} + 1}{t + 1}$$

$L =$ 

$\Delta(L) = 1 + ts$

Back to $P(K_r)$

Wedderburn's Thm: A finite dim'l algebra over an algebraically closed field K is a direct sum of matrix algebras $M_{n_i \times n_i}(K)$

Ex: G finite group
$A = \mathbb{C}[G]$ group ring

$$\mathbb{C}[G] \simeq \bigoplus \mathrm{End}_K(V_\lambda)$$

$$V_\lambda \in \mathrm{Irrep}(G)$$

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} \mathrm{End}(S_{(\lambda)})$$

Thm: $H_n \otimes \mathbb{C}(q) \simeq \bigoplus_{\lambda \vdash n} M_{n_\lambda \times n_\lambda}(\mathbb{C}(q))$

$$n_\lambda = \dim S_{(\lambda)}$$

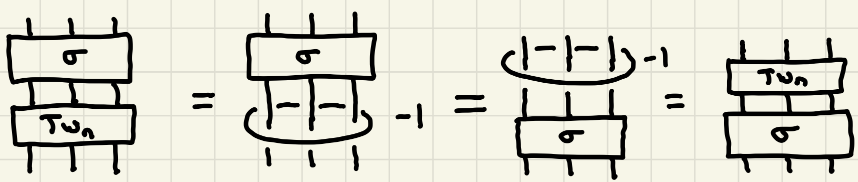
$\Rightarrow$ Center $C(H_n) = \langle e_\lambda \mid \lambda \vdash n \rangle$ $\quad e_\lambda = I \in M_{n_\lambda \times n_\lambda}$

$$e_\lambda e_\mu = \begin{cases} e_\lambda & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$$

e_λ = Jones-Wenzl projector or central idempotent.

Lemma: $Tw_n \in C(Br_n)$

Proof:



Cor: $T := \psi(T_{w_n}) \in C(H_n)$

Proof: $T_i T = \psi(\sigma_i T_{w_n}) = \psi(T_{w_n} \sigma_i) = T T_i$

$$\Rightarrow T = \sum_{\lambda \vdash n} \alpha_\lambda e_\lambda \qquad \alpha_\lambda \in \mathbb{C}(q)$$

$\sigma \in Br_n$

$$\psi(\sigma) = 1 \cdot \psi(\sigma) = \sum_\lambda e_\lambda \psi(\sigma) = \sum_\lambda \sigma_\lambda$$

$$\Rightarrow \psi(T_{w_n}^r \sigma) = \sum_\lambda \alpha_\lambda^r \sigma_\lambda$$

$$P(K_r) \sim \sum_\lambda \alpha_\lambda^r \, \mathrm{Tr}\, \sigma_\lambda$$

Formula for $P(K_r)$!

Q: What are α_λ and e_λ?

Ex: $n = 2 \quad H_2 = \langle 1, B_1 \rangle$

$$B_1^2 = [2] B_1 \qquad e_{\substack{\square\\\square}} = B_1 / [2]$$

$$1 = e_{\square\square} + e_{\substack{\square\\\square}} \Rightarrow e_{\square\square} = 1 - B_1/[2]$$

$$T_1 = \psi(\sigma_1) = q^{-1} - B_1$$

$$T_1 \cdot B_1 = q^{-1} B_1 - [2] B_1 = -q B_1$$

$$\Rightarrow \begin{array}{l} T_1 e_{\substack{\square\\\square}} = -q e_{\substack{\square\\\square}} \\ T_1 e_{\square\square} = q^{-1} e_{\square\square} \end{array} \Rightarrow \boxed{\begin{array}{l} T e_{\substack{\square\\\square}} = q^2 e_{\substack{\square\\\square}} \\ T e_{\square\square} = q^{-2} e_{\square\square} \end{array}}$$

$(T = T_1^2)$

$(T + q)(T - q^{-1}) = 0 \Rightarrow$ eigenvalues must be $q^{-1}, -q$.

General Picture:

Ideals

$I_\lambda = \langle B_s \mid \lambda(s) \geq \lambda \rangle$ $\qquad M_\lambda \subset I_\lambda$

$\Rightarrow e_\lambda \cdot I_{>\lambda} = 0$

$I_{>\lambda} = \langle B_s \mid \lambda(s) > \lambda \rangle$ $\qquad \pi : M_\lambda \xrightarrow{\cong} I_\lambda / I_{>\lambda}$

<u>Thm</u>: (K-L):

$$\begin{tikzcd} Br_n \arrow[d] & \circlearrowright & M_\lambda \arrow[d, "q=1"] \\ S_n & \circlearrowright & \mathrm{End}(S_{(\lambda)}) = M_\lambda \otimes_R (R/(q-1)) \end{tikzcd}$$

Robinson-Schensted: $s \longrightarrow T(s)$ standard tableau

$\lambda(s) = \mathrm{shape}(T(s))$

Left K-L cell $C_T = \langle B_s \mid T(s) = T \rangle$

Action of H_n on $I_\lambda / I_{>\lambda}$ preserves C_T

$$\begin{tikzcd} Br_n \arrow[d] & \circlearrowright & C_T \arrow[d, "q=1"] \\ S_n & \circlearrowright & S_{(\lambda)} \end{tikzcd}$$ commutes

<u>Ex</u>: $T = \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$ $\qquad C_T = \langle B_2, B_{12} \rangle$

$T_1 \cdot B_2 = q^{-1} B_2 - B_{12}$

$T_1 \cdot B_{12} = -q B_{12}$

$$\begin{tikzcd}[column sep=small] S_{\boxplus} = \langle & e_1 - e_2, \arrow[d, leftrightarrow] & e_1 - e_3 \arrow[d, leftrightarrow] & \rangle \\ & B_{12} & B_2 & \end{tikzcd}$$

Action of T on H_n is diagonalizable, eigenvalues α_λ.

What are the α_λ?

$T_i e_\lambda \in M_{n_\lambda \times n_\lambda}$ has eigenvalues $-q$ w/ multiplicity n_λ^+
q^{-1} w/ multiplicity n_λ^-

All T_i are conjugate, so multiplicities are independent of i

Prop (Jones): $\alpha_\lambda = q^{c_\lambda}$ with $c_\lambda = n(n-1)\frac{n^+ - n^-}{n^+ + n^-}$

$\longrightarrow P(K_r) = \sum_\lambda q^{r c_\lambda} \operatorname{Tr} \sigma_\lambda$

Proof:

$Te_\lambda = \alpha_\lambda I \in M_{n_\lambda \times n_\lambda}$ $\quad n_\lambda = \dim V_\lambda = n^+ + n^-$

$\Rightarrow \det Te_\lambda = \alpha_\lambda^{n_1 + n_2}$

$\det T_i e_\lambda = (-q)^{n_+}(q^{-1})^{n_-}$

$T = (T_1 \cdots T_{n-1})^n \Rightarrow \det T = q^{n(n-1)(n^+ - n^-)} = \alpha_\lambda^{n^+ + n^-}$ $\square$

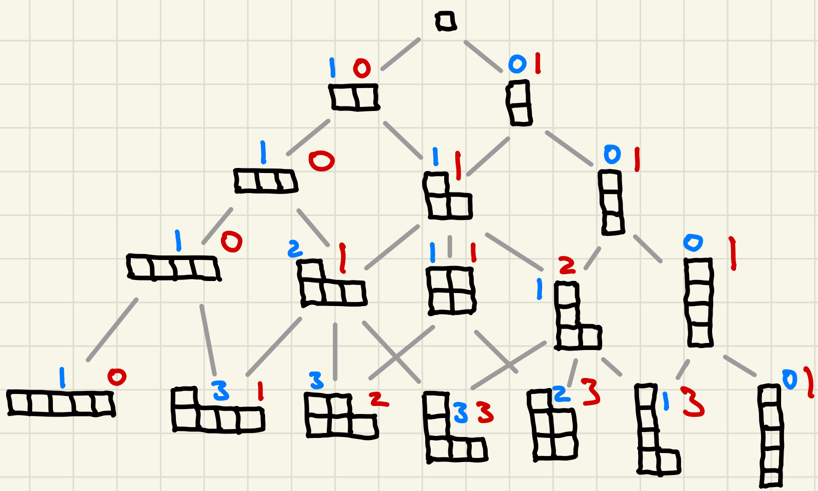
How to compute:

$$\mathrm{Res}^{H_{n-1}}_{H_n} V_\lambda = \bigoplus_{\lambda \leftarrow \lambda'} V_{\lambda'}$$

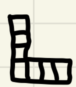
$\lambda \leftarrow \lambda'$ if λ' is obtained from λ by adding a box

n_+

n_-



c_λ	-20	-10	-4	0	4	10	20

Only hooks  contribute to Δ

Another formula

$$\kappa(\lambda) = \sum \binom{\lambda_i}{2}$$

λ' = transpose partition

$$c_\lambda = 2\left(\kappa(\lambda) - \kappa(\lambda')\right)$$

Categorification:

Recall: $SBim_n$ = graded monoidal additive category

generated by $\mathbb{B}_s$ $s \in S_n$ $\quad K(SBim_n) \simeq H_n$

$$[\mathbb{B}_s] \mapsto B_s$$

$\mathcal{P}(HH_*(\mathbb{B}_s)) = \operatorname{Tr} B_s$

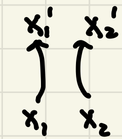
$n=2$: $\quad R = \mathbb{C}[x_1, x_2]$, $R' = \mathbb{C}[x_1, x_2, x_1', x_2']$

Picture:

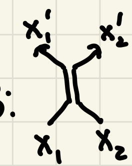
Objects:

$$\boxed{\begin{array}{l} \mathbb{1} = R = R'/(x_1 = x_1',\ x_2 = x_2') \\ \mathbb{B} = \mathbb{B}_1 = R'/\begin{pmatrix} x_1 + x_2 = x_1' + x_2' \\ x_1 x_2 = x_1' x_2' \end{pmatrix} \end{array}}$$

$$= \mathbb{C}[x_1, x_2, x_1']/(x_1' - x_1)(x_1' - x_2)$$

$\mathbb{1}$: 

$x_1 = x_1'$
$x_2 = x_2'$

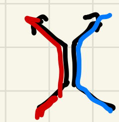
$\mathbb{B}$: 

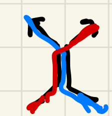
$x_1 = x_1'$
$x_2 = x_2'$
or
$x_1 = x_2'$
$x_2 = x_1'$

Homs: $\quad \operatorname{Hom}(\mathbb{B}_s, \mathbb{B}_{s'})$ is R-R bimodule

$$\boxed{\begin{array}{ll} \operatorname{Hom}(\mathbb{1},\mathbb{1}) \simeq R \cdot \mathrm{id}_1 & \operatorname{Hom}(\mathbb{B},\mathbb{1}) \simeq R \cdot x_2 \\ \operatorname{Hom}(\mathbb{1},\mathbb{B}_1) \simeq R \cdot x_1 & \operatorname{Hom}(\mathbb{B},\mathbb{B}) \simeq \mathbb{B} \cdot \mathrm{id}_{\mathbb{B}} \end{array}}$$

$x_2 : \mathbb{B} \to \mathbb{1}$
$\quad 1 \to 1$





$x_1 : R \to \mathbb{B}'$
$\quad 1 \mapsto x_1' - x_2$

Rouquier Complex:

$\psi: Br_n \to H_n$

$\sigma_i \to q^{-1} - B_i$

$\sigma_i^{-1} \to q - B_i$

$C(\sigma\tau) = C(\sigma) \otimes C(\tau)$

$C: Br_n \to Kom(SBim_n)$

$C(\sigma_i) = \boxed{q^{-1}R} \xrightarrow{x_i} \mathbb{B}_i$

$C(\sigma_i^{-1}) = \mathbb{B}_i \xrightarrow{x_2} \boxed{qR}$

in $Kom(SBim_n)$

$\boxed{}$'ed terms in homological grading 0

<u>Thm (Rouquier)</u>: This is a well-defined map

$$Br_n \to Kom(SBim_n)$$

$\tilde{C}(\sigma_i) = R \to qB_i$

$\tilde{C}(\sigma_i^{-1}) = q^{-1}B_i \to R$

$C(\sigma) = q^{-w(\sigma)}\tilde{C}(\sigma)$

$$\chi(C(\sigma)) \sim \psi(\sigma)$$

<u>Khovanov</u>: $H_*(HH^*(C(\sigma)))$ is an invariant of $\bar{\sigma}$

$$\chi(H(\bar{\sigma})) = \chi(HH^*(C(\sigma))) = HH^*(\chi(C(\sigma))) \sim HH^*(\psi(\sigma))$$
$$\sim Tr(\psi(\sigma))$$
$$\sim P(\bar{\sigma})$$

Simplify $C(\sigma)$ using

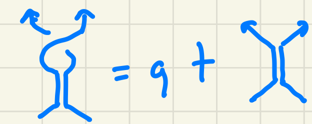
Cancellation: If $c: A \to A$ is an $\simeq$

$$C_{n+2} \xrightarrow{\begin{bmatrix} d_{n+2} \\ f\end{bmatrix}} \begin{matrix} C_{n+1} \\ \oplus \\ A\end{matrix} \xrightarrow{\begin{bmatrix}\alpha & \beta \\ \gamma & c\end{bmatrix}} \begin{matrix} C_n \\ \oplus \\ A \end{matrix} \xrightarrow{[d_n\ g]} C_{n-1} \quad \sim \quad C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{\alpha - \overbrace{\beta c^{-1}\gamma}^{\text{zig-zags}}} C_n \xrightarrow{d_{n-1}} C_{n-1}$$

Basic Lemma:

$$\tilde{C}(\sigma_1)\otimes B_1 = (\boxed{R} \to qB_1)\otimes B_1 = \boxed{B_1} \to B_1 \otimes q^2 B_1$$
$$\sim \boxed{0} \to q^2 B_1$$

$$\tilde{C}(\sigma_1^{-1})\otimes B_1 = (q^{-1}B \to \boxed{R})\otimes B_1 = q^{-2}B_1\otimes B_1 \to \boxed{B_1}$$
$$\sim q^{-2}B_1 \to \boxed{0}$$



RII: $\tilde{C}(\sigma_1)\otimes \tilde{C}(\sigma_1^{-1}) = (R \to qB_1)\otimes(q^{-1}B_1 \to R)$

$$\begin{tikzcd} B_1^2 \arrow[r] & qB_1 \\ q^{-1}B_1 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd} = \begin{tikzcd} q^{-1}+qB_1 \arrow[r] & qB_1 \\ q^{-1}B_1 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd} \sim \begin{tikzcd} q^{-1}B_1 & \\ q^{-1}B_1 \arrow[u] \arrow[r] & R \end{tikzcd} \sim R$$

<u>Thm</u>: (Soergel) $\mathrm{Hom}(B_s, B_{s'})$ is graded by degree of polynomials

$$\boxed{\mathrm{Hom}_i(B_s, B_{s'}) = \begin{cases} 0 & i<0 \\ 0 & i=0,\ s\neq s' \\ \mathbb{C}\cdot \mathrm{id}_s & i=0\ \ s=s' \end{cases}}$$

(B_s's are an exceptional collection)

<u>Def</u>: A complex C over a graded category is <u>minimal</u> if all components of d have positive degree

<u>Thm</u> $\Rightarrow$ ① Any cx over $SBim_n$ is $\sim$ to a minimal cx

② If C_1 and C_2 are minimal and $C_1 \sim C_2$, then $C_1 \simeq C_2$

Krull-Schmidt property

③ If $\bigoplus n_s B_s \simeq \bigoplus m_s B_s$

then $n_s = m_s$